Reissner-Nordström ブラックホールの状態方程式のどこに量子重力の姿が見えるか？

What is the feature of underlying quantum gravity in the equation of states of Reissner-Nordström black hole ?

斉田浩見 *
Hiromi Saida

Summary

The concrete form of equation of states, for laboratory system, is not determined by solely thermodynamics, but should be determined with the other physics such as statistical mechanics, molecular dynamics and so on. In black hole thermodynamics, the concrete form of equation of states is usually derived with some candidate theory of quantum gravity (e.g. Euclidean quantum gravity and so on). This article deals with, as an example, the equation of states of Reissner-Nordström black hole. We do not refer to any candidate theory of quantum gravity, but introduce minimal basic requirements from which the concrete form of equation of states is derived. This implies that those basic requirements are the items which should be justified by the underlying quantum gravity. That is, those basic requirements are the feature of underlying quantum gravity found in the equation of states.

キーワード：ブラックホール熱力学，状態方程式，量子重力
Keywords：Black hole thermodynamics, equation of states, quantum gravity

1　Introduction

According to the theory of general relativity, black hole is the spacetime region governed by extremely strong gravity. Nothing can escape from inside of black hole due to the strong gravity. Further, anything inside black hole can not stay at a constant (proper) distance from the center of black hole region, but is going to fall to the center, thus the black hole is an empty region [1].

In the sense of laboratory physics, we do not regard an empty region, solely, as a system which forms some physical state such as thermal equilibrium state. However, Hawking radiation is an astonishing theoretical prediction indicating that a thermal radiation is emitted from black hole horizon (surface of the empty black hole region) [2]. Black hole have to be regarded as some thermal equilibrium system. Then the fact, that Hawking radiation is the effect of quantum field theory in black hole spacetime, implies that the black hole region is a thermal equilibrium state of underlying quantum gravity. Hence, it is expected that studying thermal property of black hole is searching for some property of underlying quantum gravity.

Here note that, in laboratory thermodynamics, the concrete form of equation of states can not be determined by solely thermodynamics, but should be determined with the other physics such as statistical mechanics, molecular dynamics and so on. Then, for black holes, the concrete form of equation of states should not be determined by solely thermodynamic property of black holes, but should be determined with underlying quantum gravity (and statistical mechanics of long range interaction systems).

This article deals with, as an example, the equation of states of Reissner-Nordström black hole. We do not refer to any candidate theory of quantum gravity, but introduce minimal basic requirements from which the concrete form of equation of states is derived so as to match with the Hawking radiation and the so-called black hole's mechanical 1st and 2nd laws. This implies that those basic requirements are the minimal items which should be justified by the underlying quantum gravity. That is, those basic requirements are the feature of underlying quantum gravity found in the equation of states of Reissner-Nordström black hole.

*大同大学教養部物理学教室

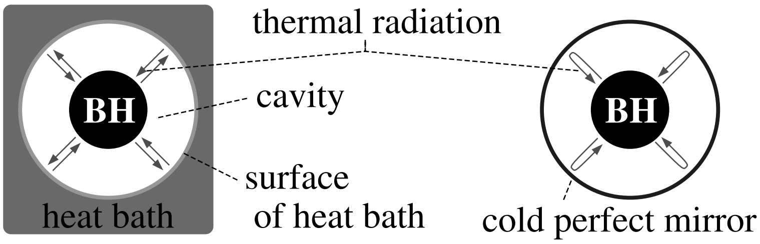

Figure 1: Two types of thermal equilibrium system of black hole

2 Equation of states for Reissner-Nordström black hole

In order to search for the basic requirements of equation of states, we need to consider a thermal equilibrium system including black hole, since, exactly speaking, the equation of states is a relation among state variables of thermal equilibrium states. We introduce a thermal equilibrium system of black hole proposed by York [3]. (Left panel in fig.1.) It is constructed by placing a black hole at the center of spherical cavity hollowed in heat bath, and adjusting the temperature of heat bath to the same value with the temperature of Hawking radiation. The Hawking radiation is completely absorbed by heat bath, and the heat bath emits a thermal radiation of the same temperature with Hawking radiation. Then, because the energy loss of black hole due to Hawking radiation is exactly balanced with the energy injection due to thermal radiation emitted from heat bath, the total system composed of black hole and thermal radiation filling the cavity becomes a thermal equilibrium system. The work and heat operate on this total system through the surface of heat bath. Since this total system contacts with a heat bath, isothermal process is possible for this total system, but adiabatic process is not possible.

Next, we find another type of thermal equilibrium system of black hole. (Right panel in fig.1.) It is constructed by placing a black hole at the center of spherical mirror whose temperature is zero and which reflects perfectly the Hawking radiation [4]. We call this mirror the *cold perfect mirror*. The cold perfect mirror emits no radiation because of zero temperature, but reflects Hawking radiation without breaking thermal spectrum of radiation. Then, since the energy loss of black hole due to Hawking radiation is exactly balanced with the energy injection due to reflected radiation, the total system composed of black hole and thermal radiation filling the space enclosed by mirror becomes a thermal equilibrium system †. Note that, the cold perfect mirror is regarded as a kind of heat insulating wall for this total system. Thermodynamic process possible for this total system is adiabatic process.

These examples are the theoretical evidence for the existence of thermal equilibrium system of black hole, i.e. the 0th law of black hole thermodynamics. Note that thermal equilibrium states of systems shown in fig.1 can be thermally stable and unstable [3, 5]. However, thermal stability is the property which should be recognized after the equation of states is specified. Thus, for the purpose of this article (search for basic requirements in the derivation of equation of states of black hole), the point is the existence of thermal equilibrium states.

In this article, we make a reasonable proposal of concrete form equations of states of Reissner-Nordström black hole, not of thermal radiation surrounding black hole because our main interest is in black hole. Note that the static coordinate (ct, r, θ, φ) is suitable to describe these systems, since thermal equilibrium state is static. When the back reaction of Hawking radiation to spacetime is negligible (as is usually the case because Hawking temperature is very low), the line element of spacetime, inside of cavity or cold perfect mirror, is

$$\mathrm{d}s^2 = -f(r)\, c^2 \mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\varphi^2 \right) \quad , \quad f(r) := 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \,, \tag{2.1a}$$

where c is the light speed, r is the areal radius, and the parameters M and Q are respectively the mass and electric charge of black hole measured in dimension of length (i.e. $M := G M_{\mathrm{BH}}/c^2$ and $Q := G^{1/2} Q_{\mathrm{BH}}/c^2$,

†One may worry about the Schwinger effect (spontaneous creation of charged particles in strong electric field) which decreases the charge of black hole. In such case, let us add the charge injection which compensates the loss of black hole charge due to Schwinger effect.

where $M_{\rm BH}$ and $Q_{\rm BH}$ are the mass and electric charge in cgs-Gauss units, G is Newton's constant). The radius of black hole horizon is,

$$r_+ = M + \sqrt{M^2 - Q^2}\,, \tag{2.1b}$$

which satisfies $f(r_+) = 0$, where $0 \le |Q| < M$ for the existence of black hole. Let $r_{\rm w}$ be the radius at the cold perfect mirror or surface of heat bath. To realize thermal equilibrium systems shown in fig.1, we require $r_+ < r_{\rm w}$ hereafter. We call $r_{\rm w}$ the *edge of our system*.

The control parameters of thermodynamic system of black hole[‡] is $(M, Q, r_{\rm w})$. Since these control parameters have the dimension of length, the basic scaling of our thermodynamic system is expressed by the scaling of length size,

$$M \to \lambda M \quad , \quad Q \to \lambda Q \quad , \quad r_{\rm w} \to \lambda r_{\rm w}\,, \tag{2.2}$$

where $\lambda\,(>0)$ is the rate of scaling.

The values of control parameters, $(M, Q, r_{\rm w})$, are determined for each thermal equilibrium state of the system shown in fig.1. However, this evidence solely can not let us decide whether these parameters themselves are state variables or not, because state variables should satisfy some suitable properties such as the classification into two categories, extensive and intensive variables. In order to specify state variables of black hole, we must specify the suitable properties of state variables. Hence, in following discussions, we construct the concrete form of equation of states of Reissner-Nordström black hole, with clarifying the basic requirements and properties of state variables.

For the first, we propose the concrete form of temperature. The temperature of black hole should be the same with that of Hawking radiation, since black hole and Hawking radiation are in thermal equilibrium. At the edge of thermodynamic system of black hole ($r = r_{\rm w}$), the temperature of Hawking radiation should be consistent with the gravitational redshift which Hawking radiation receives during propagating from the black hole horizon to edge of system. Hence the temperature, $T_{\rm BH}$, of thermal equilibrium system of black hole is

$$T_{\rm BH} = \frac{\kappa_+}{2\pi\sqrt{f_{\rm w}}} \quad , \quad \kappa_+ = \frac{1-(Q/r_+)^2}{2r_+}\,, \tag{2.3}$$

where $f_{\rm w} := f(r_{\rm w})$ and κ_+ is the surface gravity at black hole horizon. Here, $1/\sqrt{f_{\rm w}}$ is the so-called Tolman factor expressing the gravitational redshift [6].

In order to propose the other state variables of black hole, we refer to the mass formula of black hole [7] (or Iyer-Wald relation for Noether charge [8, 9]) for the case $r_{\rm w} \to \infty$,

$$\text{Differential form} \;:\; dM = \frac{\kappa_+}{2\pi}\, d\Big[\frac{A_+}{4}\Big] + \frac{1}{2r_+}\, d\big[Q^2\big] \tag{2.4a}$$

$$\text{Integrated form} \;:\; M = 2\,\frac{\kappa_+}{2\pi}\,\frac{A_+}{4} + 2\,\frac{1}{2r_+}\,Q^2\,, \tag{2.4b}$$

where $A_+ := 4\pi r_+^2$ is the surface area of black hole horizon. It is usual to regard eq.(2.4a) as the limit ($r_{\rm w} \to \infty$) of differential form of 1st law for black hole state variables. The points of this formula are summarized as follows:

(P1) Mass, M, in left-hand side comes from the Noether charge at spatial infinity.

(P2) Factor, $\kappa_+/(2\pi)$, in right-hand side of eq.(2.4a) is $T_{\rm BH}$ as $r_{\rm w} \to \infty$.

(P3) Factor, $A_+/4$, in right-hand side comes from the Noether charge at horizon.

(P4) Second term in right-hand side of eq.(2.4b), $r_+^{-1} Q^2$, comes from the energy of electro-magnetic field (infinite-volume integral of time-time component of stress-energy tensor). The contents of this term is as follows,

(P4-1) Factor Q^2 is simply a constant prefactor of the infinite-volume integral.

(P4-2) Factor $1/r_+$ is the result of infinite-volume integration.

[‡]When we consider the possibility that the system can be in non-equilibrium state, we use the term *thermodynamic system*. When we emphasize that the system is in thermal equilibrium state, we use the term *thermal equilibrium system*.

From the point (P1), we require that the internal energy of black hole, $E_{\rm BH}$, satisfies,

$$E_{\rm BH} \to M \quad \text{as} \quad r_{\rm w} \to \infty \,, \tag{2.5}$$

where the limit $r_{\rm w} \to \infty$ is assigned, because the Noether charge at infinity is the origin of M in eq.(2.4).

From the points (P2) and (P3), the first term, $(\kappa_+/2\pi)\mathrm{d}[A_+/4]$, in right-hand side of eq.(2.4a) seems to be the heat term. Then, we require that the entropy of black hole, $S_{\rm BH}$, is given by

$$S_{\rm BH} = \pi r_+^2 \; (= A_+/4) \,, \tag{2.6}$$

which hold even for finite $r_{\rm w}$, since the Noether charge at horizon seems to be independent of $r_{\rm w}$. This is the so-called *entropy-area law* of single black hole, proposed originally by Bekenstein [10, 11].

From the point (P4), the second term in right-hand side of eq.(2.4a), $(2r_+)^{-1}\mathrm{d}[Q^2]$, corresponds to the variation of electro-magnetic energy. Therefore, it is reasonable to regard this term not as the work term but as the "chemical" term which describes the variation of energy due to variation of amount of matter (electro-magnetic field). Then, we require that the state variable which corresponds to "mol number", $N_{\rm BH}$, is given by the "prefactor of energy integral" explained in (P4-1),

$$N_{\rm BH} = Q^2 \,. \tag{2.7}$$

This form should hold even for finite $r_{\rm w}$, since the prefactor of energy integral (a volume integral of time-time component of stress-energy tensor at fixed time) seems to be independent of $r_{\rm w}$. And, further, we require the state variable which corresponds to "chemical potential", $\mu_{\rm BH}$, satisfies,

$$\mu_{\rm BH} \to \frac{1}{2r_+} \quad \text{as} \quad r_{\rm w} \to \infty \,, \tag{2.8}$$

where the limit $r_{\rm w} \to \infty$ is assigned, because the infinite-volume integral of stress-energy tensor is the origin of the factor $1/2r_+$ in "chemical" term of eq.(2.4a).

From the above, we can infer a peculiar property of state variables of black hole [3, 4]:

Requirement 1 (classification of black hole's state variables) *Under the basic scaling of length size such as eq.*(2.2)*, all state variables of black hole are classified into three categories according to scaling behavior;*

- **BH-extensive state variables:** *These state variables, $\Phi^{(\rm ex)}_{\rm BH}$ (e.g. $S_{\rm BH}$ and $N_{\rm BH}$), are scaled by the dimension of area, $\Phi^{(\rm ex)}_{\rm BH} \to \lambda^2 \Phi^{(\rm ex)}_{\rm BH}$.*

- **BH-intensive state variables:** *These state variables, $\Phi^{(\rm in)}_{\rm BH}$ (e.g. $T_{\rm BH}$ and $\mu_{\rm BH}$), are scaled by the dimension of inverse of length, $\Phi^{(\rm ex)}_{\rm BH} \to \lambda^{-1} \Phi^{(\rm ex)}_{\rm BH}$.*

- **BH-energy state variables:** *These state variables, $\Phi^{(\rm ene)}_{\rm BH}$ (e.g. $E_{\rm BH}$), are scaled by the dimension of length, $\Phi^{(\rm ene)}_{\rm BH} \to \lambda \Phi^{(\rm ene)}_{\rm BH}$.*

This classification is different from that of laboratory thermodynamics. □

Here remember the classification of state variables for laboratory systems:

- Extensive variables for laboratory systems are scaled with the same dimension of system size: Under the scaling of system size (e.g. volume for a gas in a box, area for a labor sheet) $V \to \alpha V$, the extensive variables, X, of laboratory systems scales as $X \to \alpha X$.

- Intensive variables for laboratory systems are invariant under the scaling of system size.

- The energy state variables in laboratory systems is degenerated to extensive variables.

Hence, it is obvious that the classification of state variables should be modified in black hole thermodynamics.

Given the scaling behavior of BH-extensive state variables, we can not regard the three dimensional volume $\int_{r_+}^{r_{\rm w}} {\rm d}r\, 4\pi r^2/\sqrt{f(r)}$ as a system size for finite $r_{\rm w}$. However, since the system size should be a BH-extensive state variable, it is reasonable to regard the area of edge of system, $4\pi r_{\rm w}^2$, as the system size [3]. Therefore, we require that the size of thermal equilibrium system of black hole, $A_{\rm BH}$, is given by

$$A_{\rm BH} \,=\, 4\pi r_{\rm w}^2 \,. \tag{2.9}$$

Then, the complete set of system sizes is

$$X_{\rm BH} \,=\, (A_{\rm BH}, N_{\rm BH}) \,. \tag{2.10}$$

In laboratory thermodynamics of a gas in a box, the counterpart to this $X_{\rm BH}$ is the set of volume and mol number.

Next, let us propose the forms of internal energy, $E_{\rm BH}$, and of free energy, $F_{\rm BH}$. Note that, in York's derivation of them, Euclidean quantum gravity has been used [3, 5]. However, we prefer deriving state variables of black hole without using any existing candidate theory of quantum gravity (such as Euclidean quantum gravity, super string theory, loop quantum gravity, causal dynamical triangulation and so on), because, as discussed in sec.1 and previous paper [4], we expect black hole thermodynamics to be the clue to some universal property of quantum gravity which is independent of any existing candidate of quantum gravity. Therefore, in following discussion, we derive $E_{\rm BH}$ and $F_{\rm BH}$ without using any candidate of quantum gravity.

As in laboratory thermodynamics, we require that the internal energy is a function of entropy and system size, $E_{\rm BH}(S_{\rm BH}, A_{\rm BH}, N_{\rm BH})$, and the free energy is a function of temperature and system size, $F_{\rm BH}(T_{\rm BH}, A_{\rm BH}, N_{\rm BH})$. Furthermore, as in laboratory thermodynamics, we require the differential relation,

$$\frac{\partial F_{\rm BH}(T_{\rm BH}, A_{\rm BH}, N_{\rm BH})}{\partial T_{\rm BH}} \,=\, -S_{\rm BH} \,. \tag{2.11a}$$

Here, transform the independent control parameters from $(M, Q, r_{\rm w})$ to $(r_+, Q, r_{\rm w})$, and regard $F_{\rm BH}$, $T_{\rm BH}$ and $S_{\rm BH}$ as functions of these parameters. Then, eq.(2.11a) becomes

$$\frac{\partial F_{\rm BH}(r_+, Q, r_{\rm w})}{\partial r_+} \,=\, -\pi r_+^2\, \frac{\partial T_{\rm BH}(r_+, Q, r_{\rm w})}{\partial r_+} \,, \tag{2.11b}$$

which gives,

$$F_{\rm BH}(r_+, Q, r_{\rm w}) \,=\, -\pi \int r_+^2\, \frac{\partial T_{\rm BH}}{\partial r_+}\, {\rm d}r_+ \,=\, -\pi r_+^2 T_{\rm BH} - r_{\rm w}\sqrt{f_{\rm w}} + h(Q, r_{\rm w}) \,, \tag{2.11c}$$

where a partial integral and eq.(2.3) are used, and $h(Q, r_{\rm w})$ is an arbitrary function of $(Q, r_{\rm w})$. By the scaling behavior of BH-energy state variables in requirement 1, we can set

$$h(Q, r_{\rm w}) \,=\, a\, r_{\rm w} + Q\, h_1\Bigl(\frac{Q}{r_{\rm w}}\Bigr) \,, \tag{2.11d}$$

where $h_1(x)$ is an arbitrary function of single argument x, and a is a constant. The term $a r_{\rm w}$ can be included into the term $Q h_1$, but we separate them for later use. As in laboratory thermodynamics, we require the Legendre transformation between $E_{\rm BH}$ and $F_{\rm BH}$, which is $E_{\rm BH} = F_{\rm BH} + T_{\rm BH}\, S_{\rm BH}$. Expressing this relation with parameters $(r_+, Q, r_{\rm w})$, we find

$$E_{\rm BH}(r_+, Q, r_{\rm w}) \,=\, -r_{\rm w}\sqrt{f_{\rm w}} + a\, r_{\rm w} + Q\, h_1\Bigl(\frac{Q}{r_{\rm w}}\Bigr) \,. \tag{2.11e}$$

Then, the requirement in eq.(2.5) results in $a = 1$ and a restriction on h_1,

$$h_1\Bigl(\frac{Q}{r_{\rm w}}\Bigr) \,=\, 0 \quad \text{or} \quad O\Bigl(\frac{Q}{r_{\rm w}}\Bigr)^{\varepsilon} \,,\ \varepsilon > 0 \,. \tag{2.11f}$$

Here let us introduce the other physical requirement from the extremal black hole limit (maximal charge limit), $Q \to M$. Eq.(2.3) gives the limit $T_{\rm BH} \to 0$ as $Q \to M$. This implies that thermal contact between black hole and heat bath (or cold perfect mirror) at $r_{\rm w}$ disappears as $Q \to M$, since temperature vanishes $T_{\rm BH} \to 0$. Then, we require that

$$\text{the dependence of } E_{\rm BH} \text{ on } r_{\rm w} \text{ disappears as } Q \to M . \tag{2.12}$$

By this requirement, we choose $h_1 = 0$ from eq.(2.11f). Hence, we obtain

$$E_{\rm BH} = r_{\rm w}\left(1-\sqrt{f_{\rm w}}\right) \tag{2.13}$$

$$F_{\rm BH} = r_{\rm w}\left(1-\sqrt{f_{\rm w}}\right) - \frac{r_+}{4\sqrt{f_{\rm w}}}\left(1-\frac{Q^2}{r_+^2}\right) . \tag{2.14}$$

Obviously, these energies satisfy the scaling behavior of BH-energy state variables given in requirement 1. Note that this form of internal energy (and free energy) is the same with that obtained by York's group using Euclidean quantum gravity [3, 5]. That is, it is not necessarily needed to refer to candidates of quantum gravity when we search for an appropriate form of internal energy.

Next, in order to propose the form of chemical potential, $\mu_{\rm BH}$, we require the differential relation as in laboratory thermodynamics,

$$\frac{\partial E_{\rm BH}(S_{\rm BH}, A_{\rm BH}, N_{\rm BH})}{\partial N_{\rm BH}} = \mu_{\rm BH} . \tag{2.15}$$

Expressing this relation with parameters $(r_+, Q, r_{\rm w})$, we find

$$\mu_{\rm BH} = \frac{1}{2Q}\,\frac{\partial E_{\rm BH}(r_+, Q, r_{\rm w})}{\partial Q} = \frac{1}{2\sqrt{f_{\rm w}}}\left(\frac{1}{r_+} - \frac{1}{r_{\rm w}}\right) . \tag{2.16}$$

Obviously, $\mu_{\rm BH}$ satisfies the scaling behavior of BH-intensive state variables given in requirement BH 1. Note that this $\mu_{\rm BH}$ is consistent with requirement in eq.(2.8).

As for the pressure in laboratory thermodynamics, let us define the BH-extensive state variable which is thermodynamically conjugate to the system size $A_{\rm BH}$ by

$$\begin{aligned} \sigma_{\rm BH} &:= -\frac{\partial E_{\rm BH}(S_{\rm BH}, A_{\rm BH}, N_{\rm BH})}{\partial A_{\rm BH}} = -\frac{1}{8\pi r_{\rm w}}\,\frac{\partial E_{\rm BH}(r_+, r_{\rm w}, Q)}{\partial r_{\rm w}} \\ &= \frac{1}{8\pi r_{\rm w}}\left[-1+\sqrt{f_{\rm w}} + \frac{r_+}{2r_{\rm w}\sqrt{f_{\rm w}}}\left\{1+\frac{Q^2}{r_+^2}\left(1-2\frac{r_+}{r_{\rm w}}\right)\right\}\right] . \end{aligned} \tag{2.17}$$

Obviously, this satisfies the scaling behavior of BH-extensive state variables given in requirement BH 1. We find a limiting behavior, $\sigma_{\rm BH} A_{\rm BH} \to 0$ as $r_{\rm w} \to \infty$. This behavior is useful for next paragraph. (There are some other notes on thermodynamic meaning of $\sigma_{\rm BH}$ which are found in appendix B of previous paper [12]. But such detail is not used in this paper.)

From eq.(2.3) and (2.13), we find a differential relation which is suitable as thermodynamics,

$$\frac{\partial E_{\rm BH}(S_{\rm BH}, A_{\rm BH}, N_{\rm BH})}{\partial S_{\rm BH}} = T_{\rm BH} . \tag{2.18}$$

Then, the differential form of 1st law is obtained,

$$dE_{\rm BH}(S_{\rm BH}, A_{\rm BH}, N_{\rm BH}) = T_{\rm BH}\,dS_{\rm BH} - \sigma_{\rm BH}\,dA_{\rm BH} + \mu_{\rm BH}\,dN_{\rm BH} . \tag{2.19}$$

As $r_{\rm w} \to \infty$, we find that the work term vanishes, $\sigma_{\rm BH}\,dA_{\rm BH} \to 0$. Therefore, eq.(2.19) reproduces eq.(2.4a) at the limit $r_{\rm w} \to \infty$. Further, from concrete forms of state variables obtained above, we find,

$$E_{\rm BH} = 2T_{\rm BH}S_{\rm BH} - 2\sigma_{\rm BH}A_{\rm BH} + 2\mu_{\rm BH}N_{\rm BH} . \tag{2.20}$$

This reproduces eq.(2.4b) at the limit $r_{\rm w} \to \infty$.

So far we have constructed the concrete examples of thermal equilibrium system of black hole shown in fig.1. Also, we have proposed the concrete form of state variables for the system. Those state variables are regarded as the equations of states for the thermal equilibrium system of black hole.

Remember that, exactly speaking, our thermodynamic system of black hole is composed of black hole and matter fields of thermal radiation surrounding black hole as shown in fig.1. Therefore, thermal equilibrium state of this system is expressed as a composite system,

$$(\widetilde{\Phi}_{\rm BH}\, ,\, \widetilde{\Phi}_{\rm m})\,, \tag{2.21}$$

where $\widetilde{\Phi}_{\rm BH}$ is the set of independent state variables of black hole and $\widetilde{\Phi}_{\rm m}$ is that of matter field of radiation. Here, one may consider that, when the radiation matter is explicitly considered, the independent control parameters of our system are not only $(M, Q, r_{\rm w})$ but also some quantity of radiation matter. However, this consideration is not true of thermal radiation surrounding black hole, because the temperature and volume of thermal radiation, which are the independent state variables of radiation, are determined once the values of $(M, Q, r_{\rm w})$ are specified. (Temperature and volume are given respectively by the Hawking temperature and spatial volume between r_+ and $r_{\rm w}$.) Hence, even when we consider the composite system of black hole and thermal radiation in cavity or cold perfect mirror, the thermal equilibrium state $(\widetilde{\Phi}_{\rm BH}\, ,\, \widetilde{\Phi}_{\rm m})$ can be controlled by parameters $(M, Q, r_{\rm w})$[§].

Finally, let us note a lesson which we can learn from the concrete form of free energy (2.14). As summarized in appendix A (fig.2), for fixed system size $(A_{\rm BH}, N_{\rm BH})$, $F_{\rm BH}$ becomes single-valued function of $T_{\rm BH}$ when $A_{\rm BH}/N_{\rm BH}$ is small, but becomes many-valued function of $T_{\rm BH}$ when $A_{\rm BH}/N_{\rm BH}$ is large. Then, it is found in appendix A that thermal equilibrium states for the case of single-valued $F_{\rm BH}$ is always thermally stable (heat capacity is positive). For this case, thermal equilibrium state is uniquely determined by state variables $(T_{\rm BH}, A_{\rm BH}, N_{\rm BH})$. On the other hand, it is also found in appendix A that both stable (positive heat capacity) and unstable (negative heat capacity) thermal equilibrium states are possible for the case of many-valued $F_{\rm BH}$. For this case, thermal equilibrium state is uniquely determined by not only $(T_{\rm BH}, A_{\rm BH}, N_{\rm BH})$ but also another state variable (e.g. entropy $S_{\rm BH}$, chemical potential $\mu_{\rm BH}$, or the like). The latter state variable specifies whether the thermal equilibrium state is stable or unstable. Therefore, the lesson learned from eq.(2.14) is as follows:

(L1) For a domain of $(T_{\rm BH}, A_{\rm BH}, N_{\rm BH})$, the number of independent state variables is three, which is the same number of independent control parameters $(r_+, Q, r_{\rm w})$. In this case, for given values of $(T_{\rm BH}, A_{\rm BH}, N_{\rm BH})$, only one stable thermal equilibrium state is possible.

(L2) For the domain of $(T_{\rm BH}, A_{\rm BH}, N_{\rm BH})$ complementary to that in (L1), the number of independent state variables is not three but four. In this case, for given values of $(T_{\rm BH}, A_{\rm BH}, N_{\rm BH})$, stable and unstable thermal equilibrium states are possible. These states are distinguished by another independent state variable other than $(T_{\rm BH}, A_{\rm BH}, N_{\rm BH})$.

These lessens are understood as a basic property of thermal equilibrium state of black hole. Then, we can summarize the 0th law in black hole thermodynamics as follows:

Requirement 2 (0th law in black hole thermodynamics) *To construct the notion of thermal equilibrium state of our system, we divide this requirement into two statements:*

1st statement: *For ordinary matters (e.g. thermal radiation), the 0th law of ordinary thermodynamics holds, in which the notion of heat bath, temperature, system size and thermal equilibrium state are introduced.*

2nd statement: *When a black hole is placed in a cavity or cold perfect mirror as shown in fig.1, the black hole and thermal radiation surrounding black hole form a thermal equilibrium composite system. And, there exist temperature (Hawking temperature) and system size (surface area at the edge of system and square of charge) which determines a thermal equilibrium state of black hole as follows:*

- *There exists a domain of black hole temperature and system size, where the thermal equilibrium state is uniquely determined by the temperature and system size.*
- *When the black hole temperature and system size is not in the domain mentioned above, the thermal equilibrium state is uniquely determined by not only the temperature and system size but also another state variable (e.g. entropy, or the like).*

[§]If some matters other than radiation is included in our system, then some suitable parameter of the additional matter should be added to the set of control parameters of our system.

These statements, especially the second statement, specify the basic property of thermal equilibrium state of black hole. □

Note that the second statement in requirement 2 is based on the lessens learned from concrete form of free energy (2.14). However, in the second statement, we have not mentioned the stability of thermal equilibrium states, because the notion of heat capacity (or its equivalent statements) should be introduced in order to discuss the stability of thermal equilibrium states, and the heat capacity does not appear in 0th law.

3 Conclusion

In previous section, we have constructed the concrete forms of state variables of Reissner-Nordström black hole, The relation among state variables, i.e. the concrete form of equation of states, is obtained from those concrete forms of state variables. Sec.2 is summarized as follows:

Requirement 3 (Basic requirements for equation of states) *The basic requirements in the derivation of state variables are:*

(A) *The classification of state variables in requirement 1*

(B) *Eq.*(2.3)*:* $T_{\rm BH} = \dfrac{\kappa_+}{2\pi\sqrt{f_{\rm w}}}$

(C) *Eq.*(2.5)*:* $E_{\rm BH} \to M \quad as \quad r_{\rm w} \to \infty$

(D) *Eq.*(2.6)*:* $S_{\rm BH} = \dfrac{A_+}{4}$

(E) *Eq.*(2.7)*:* $N_{\rm BH} = Q^2$

(F) *Eq.*(2.8)*:* $\mu_{\rm BH} \to \dfrac{1}{2r_+} \quad as \quad r_{\rm w} \to \infty$

(G) *Eq.*(2.9)*:* $A_{\rm BH} = 4\pi r_{\rm w}^2$

(H) *Eq.*(2.12)*:* $\dfrac{\partial E_{\rm BH}}{\partial r_{\rm w}} \to 0 \quad as \quad Q \to M - 0$

From these eight requirements, we can obtain the concrete form of equation of states of Reissner-Nordström black hole. Then, consequently, the statement of 0th law is found as summarized in requirement 2. □

Some of these requirements may be directly the feature of underlying quantum gravity, and the others may be of simply the statistical property of long range interaction systems. The latter should be found in the statistical mechanics for long range interaction systems. Hence, a comparison of black hole thermodynamics with statistical mechanics of long range interaction system reveals which of the eight items in requirement 3 is the statistical property of black hole as a long range interaction system. The rest items should be regarded as the feature of underlying quantum gravity found in black hole thermodynamics of Reissner-Nordström black hole.

Acknowledgment

This work is supported by an inside grant of Daido university during Apr.2012 to Mar.2013.

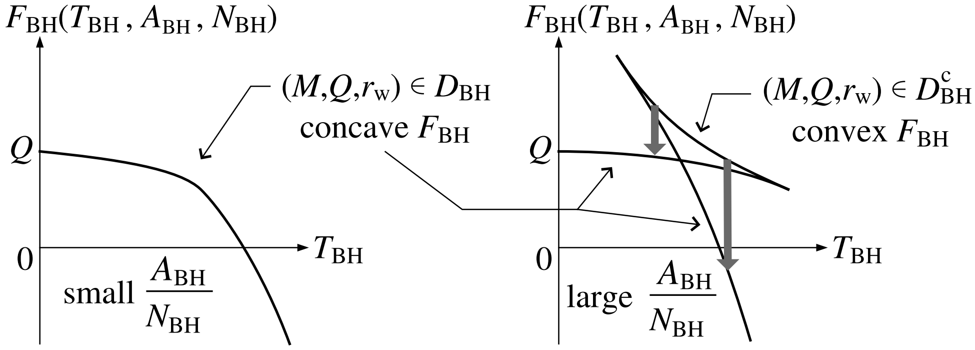

Figure 2: Schematic graphs of $F_{\rm BH}(T_{\rm BH}, A_{\rm BH}, N_{\rm BH})$ for fixed $A_{\rm BH}$ and $N_{\rm BH}$: It is found that the free energy of unstable thermal equilibrium state (negative heat capacity) is always higher than that of stable thermal equilibrium state (positive heat capacity).

A Stabilization of thermal equilibrium state of black hole in a heat bath

In this appendix, we discuss the evolution of unstable thermal equilibrium state of thermodynamic system of black hole in cavity (left panel in fig.1). Further in this appendix, we use the concrete forms of state variables suggested in sec.2. Also we refer to the fact, which will be proven in next paper, that the thermal equilibrium state of black hole system in cavity evolves towards a thermal equilibrium state of lower free energy. (This is the fact which has already been proven for laboratory systems in an environment of *constant temperature*. In next paper, this fact will be proven for thermodynamic systems of black hole.)

For complete description of thermodynamic behavior of system in the environment of constant temperature, the free energy should be regarded as a function of temperature and system size, $F_{\rm BH}(T_{\rm BH}, A_{\rm BH}, N_{\rm BH})$. The heat capacity at constant system size, $C_{\rm BH}$, is defined as

$$C_{\rm BH} := T_{\rm BH} \frac{\partial S_{\rm BH}(T_{\rm BH}, A_{\rm BH}, N_{\rm BH})}{\partial T_{\rm BH}} = -T_{\rm BH} \frac{\partial^2 F_{\rm BH}(T_{\rm BH}, A_{\rm BH}, N_{\rm BH})}{\partial T_{\rm BH}^2} . \tag{A.1}$$

This denotes that thermal equilibrium state is stable (i.e. $C_{\rm BH} > 0$), if $F_{\rm BH}$ is concave with respect to $T_{\rm BH}$ (i.e. $\partial^2 F_{\rm BH}/\partial T_{\rm BH}^2 < 0$). Also, thermal equilibrium state is unstable (i.e. $C_{\rm BH} < 0$), if $F_{\rm BH}$ is convex with respect to $T_{\rm BH}$ (i.e. $\partial^2 F_{\rm BH}/\partial T_{\rm BH}^2 > 0$).

On the other hand, from the concrete forms of $T_{\rm BH}$ in eq.(2.3) and $F_{\rm BH}$ in eq.(2.14), we obtain the graph in fig.2 which shows the behavior of $F_{\rm BH}(T_{\rm BH}, A_{\rm BH}, N_{\rm BH})$ with respect to $T_{\rm BH}$. From this graph and definition (A.1), we find that the free energy of unstable thermal equilibrium state (negative heat capacity) is always higher than that of stable thermal equilibrium state (positive heat capacity). Hence, because the system evolves to a thermal equilibrium state of lower free energy in the environment of constant temperature, an unstable thermal equilibrium state settles down to a stable thermal equilibrium state. This stabilization is shown in right panel in fig.2 by the arrow pointing downwards.

References

[1] V.P. Frolov and I.D. Novikov, *Black Hole Physics*, Kluwer Academic Publ., Netherlands 1998.

[2] S.W. Hawking, *Particle Creation by Black Holes*, Commun. Math. Phys. **43**, 199-220 (1975).

[3] J.W. York Jr., *Black-hole thermodynamics and the Euclidean Einstein action*, Phys. Rev. D, **33**, 2092-2099 (1986).

[4] H. Saida, *Universal Property of Quantum Gravity implied by Uniqueness Theorem of Bekenstein-Hawking Entropy*, Entropy, **13**, 1611-1647 (2011).

[5] H.W. Braden, J.D. Brown, B.F. Whiting and J.W. York Jr., *Charged black hole in a grand canonical ensemble*, Phys. Rev. D, **42** 3376-3385 (1990).

[6] R.C. Tolman, *Relativity, Thermodynamics and Cosmology*, Dover Publ., USA, 1987.

[7] J.M. Bardeen, B. Carter and S.W. Hawking, *The Four Laws of Black Hole Mechanics*, Commun. Math. Phys., **31**, 161-170 (1973).

[8] R.M .Wald, *Black hole entropy is the Noether charge*, Phys. Rev. D, **48** R3427-R3431 (1993).

[9] V. Iyer and R.M. Wald, *Some properties of the Noether charge and a proposal for dynamical black hole entropy*, Phys. Rev. D, **50** 846-864 (1994).

[10] J.D. Bekenstein, *Black Holes and Entropy*, Phys. Rev. D, **7**, 2333-2346 (1973).

[11] J.D. Bekenstein, *Generalized second law of thermodynamics in black-hole physics*, Phys. Rev. D, **9**, 3292-3300 (1974).

[12] H. Saida, *de Sitter Thermodynamics in the Canonical Ensemble*, Prog. Theor. Phys., **122**, 1239-1266 (2009).