Special function: Leaf function $r=sleaf_n(l)$ (First report)

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Summary

Special function: The leaf function $sleaf_n(l)$, together with some of its features, is presented. A saw-tooth wave with periodicity can be defined as a continuous function $sleaf_n(l)$. The exponent m of the function $\left(sleaf_n(l)\right)^m$ increases when differential operations are conducted. These leaf functions are closely related to trigonometric functions or the elliptic function. The inverse trigonometric and inverse elliptic functions are represented by $\int \frac{dt}{\sqrt{1-x^2}}$ and $\int \frac{dt}{\sqrt{1-x^4}}$, respectively. According to the Ref. [3], "mathematicians accepted the fact that $\int \frac{dt}{\sqrt{1-x^4}}$ is a new function, which is one of a family called the elliptic integrals". On the other hand, we have not discussed the higher order of the variable x, such as the inverse functions: $\int \frac{dt}{\sqrt{1-x^6}}$, $\int \frac{dt}{\sqrt{1-x^8}}$, and $\int \frac{dt}{\sqrt{1-x^{100}}}$ etc.

This paper presents a new special function, the leaf function, based on these inverse integral functions. Compared to the waves or curves produced by both the trigonometric functions and the elliptic function, different waves or curves with periodicity can be produced by using the leaf function.

Keywords : Leaf function, Leaf curve, Jacobi elliptic functions, Elliptic integrals, Lemniscate, Ordinary differential equation, Square root of polynomial

1. Introduction

In this paper, variables are always real numbers. Complex numbers are not considered. We discuss the following ordinary differential equation (ODE):

$$\frac{d^2r(l)}{dl^2} = -n \cdot r(l)^{2n-1} \quad (1)$$

$$r(0) = 0 \quad (2)$$

$$\frac{dr(0)}{dl} = 1 \quad (3)$$

The variable $r(l)$ represents the function with respect to the variable l. Equations (2) and (3) represent the initial conditions of the ODE. The number n represents a natural number $(n=1,2,3,\cdots)$. Ordinary differential equation (1) has interesting properties and can be solved by using numerical simulation techniques. In the graph, variables r and l are represented by the vertical and horizontal axes, respectively. With respect to any natural number n in Eq. (1), the graph shows various waves with periodicity.

In the case of $n=1$ in Eq.(1), we can obtain trigonometric functions (such as $r(l)=\sin(l)$ or $r(l)=\cos(l)$ etc.) as solutions of this equation. In the case of $n=2$ in Eq. (1), we can obtain the following:

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$$\frac{d^2r}{dl^2}=-2r^3 \tag{4}$$

Differentiating the function r generally leads to a decrease in the index n of the function r. Therefore, it is difficult to describe the function r by using elementary functions. As described later, in the case of $n=2$, Eq. (1) is closely related with elliptic function and integration. In the case of $n=3$, to the best of our knowledge, the following equation has not been discussed [1]-[10]:

$$\frac{d^2r}{dl^2}=-3r^5 \tag{5}$$

Using the graph or numerical analysis, the relation between the geometry and equation (1) is described for Eq. (1). As an application, the present paper deals with $n=1,2,3,4,5$ and 100. The leaf function $sleaf_n(l)$ satisfied with Eq. (1) - (3) is presented.

2. Symbols

The symbols used in the paper are as follows:

n: Natural number ($n=1,2,3,\cdots$). In the paper, it is named as basis.
r: Distance between the origin and the point on the curve

$$r=\sqrt{x^2+y^2}\geq 0 \tag{6}$$

As described below, the negative variable r has to be defined in Eq. (1).
θ: The variable represents the angle. In this paper, the unit is radian. Counter-clockwise is positive. Clockwise is negative.
l: Arc length on a leaf curve

Numerical values are rounded off to five decimal places, and calculated with a precision of up to four digits.

3. Leaf function

3.1 Elliptic function [1]

The incomplete elliptic integral of the first kind l is defined as:

$$l=\int_0^r \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} \quad -1\leq r\leq 1 \tag{7}$$

where the parameter k is the modulus of the elliptic integral. The inverse elliptic function $arcsn(r,k)$ is defined as follows:

$$arcsn(r,k)=\int_0^r \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} \quad -1\leq r\leq 1 \tag{8}$$

Therefore, the following is obtained:

$$r=sn(l,k) \tag{9}$$

3.2 Leaf curve (x - y plane)

The leaf curve is defined as follows:

$$r^n=\sin n\theta \quad n=1,2,3,\cdots \quad (r\geq 0) \tag{10}$$

A point on the graph of Eq. (10) starts at the origin. As the angle θ increases, the point moves farther away from the origin. After reaching $r=1.0$ (the distance between the point and the origin), the point returns to the origin. In the graph, the horizontal axis and the vertical axis are set to represent x and y, respectively. These curves on the graph resemble a leaf shape. Therefore, these curves are defined as the leaf curve.

The leaf curve of $n=1$ is shown in Fig.1. In this case, the leaf curve represents a circle. In this paper, the curves are defined as one positive leaf curve. The reason as to why in one leaf curve is defined as positive, is described later. The leaf curve of $n=2$ is shown in Fig.2. This leaf curve represents the lemniscate with a slope of 45 degrees. The leaf curve ($sleaf_n(l)$) and the straight line ($y=tan(\pi/4)\times x$) intersect at a point, which takes the maximum value $r = 1$.

The leaf curves of $n=3, 4, 5,$ and 100 are shown in Figs. 3-6, respectively. The graphs of these curves are described as three positive leaf curve, four positive leaf curve, five positive leaf curve, and hundred positive leaf curve, respectively. The leaf curve and the straight line $y=tan(\pi/2n)\times x$ intersect at a point, which takes the maximum value $r = 1$. The parameter n represents the natural number in Eq. (10). As the parameter n increases, the number of leaves increases in the graph.

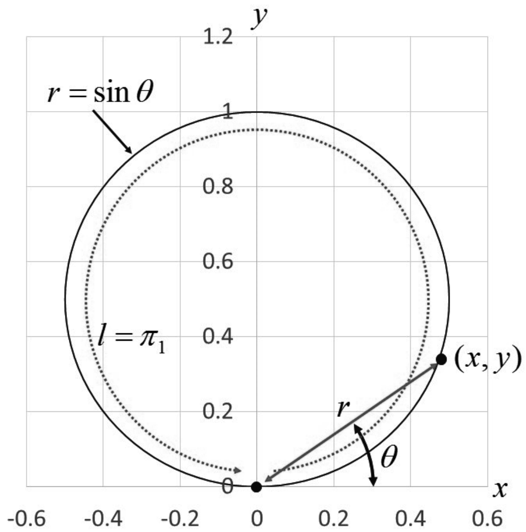


Fig. 1　　One positive leaf curve
(Circle of center (0.0, 0.5))

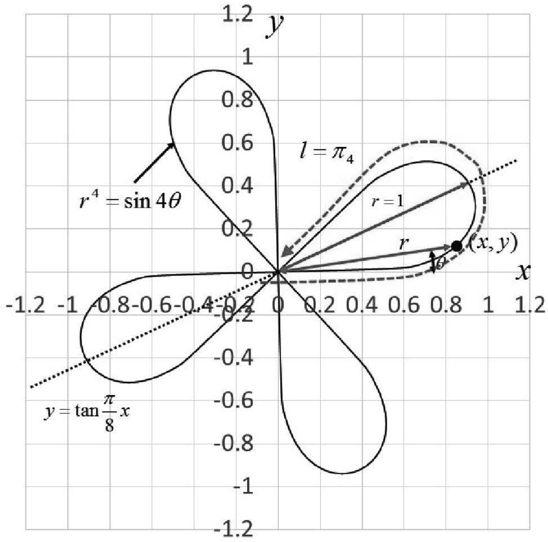


Fig. 4　　Four positive leaf curve

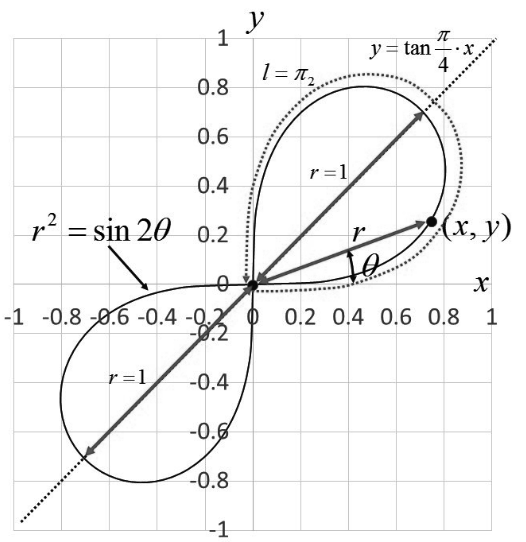


Fig. 2　　Two positive leaf curve
(lemniscate with slope of *45* degrees)

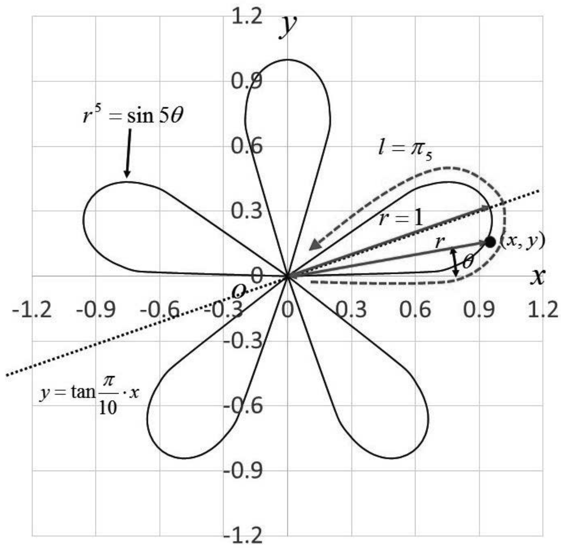


Fig. 5　　Five positive leaf curve

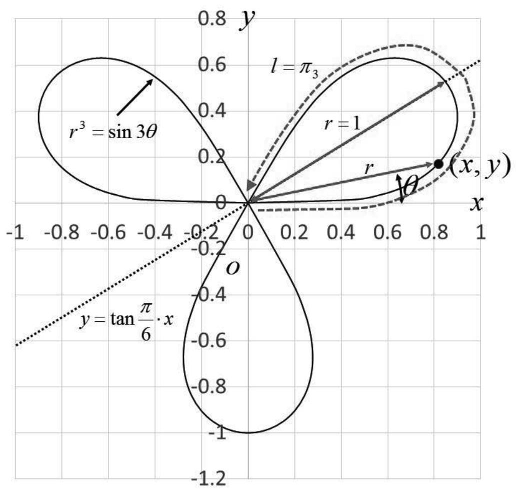


Fig. 3　　Three positive leaf curve

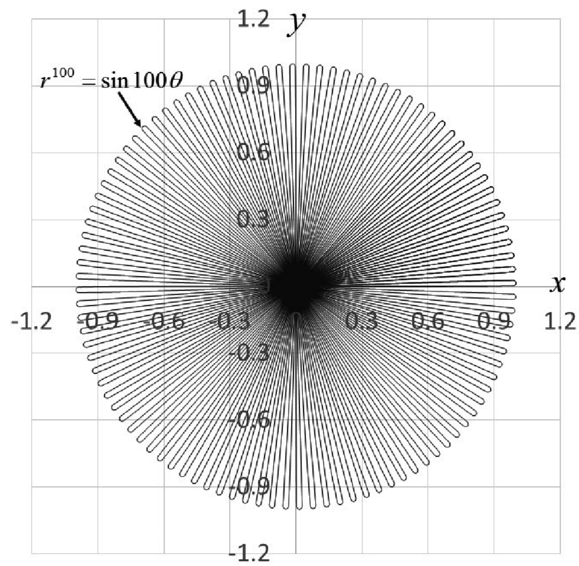


Fig. 6　　Hundred positive leaf curve

3.3 Leaf function (*r-l* plane) (in first quadrant)

In this section, we discuss the ODE in Eq. (1). The parameter *n* represents a natural number. The variable *l* represents the length between the origin and the point on the leaf curve.

For example, the cases of *n=1,2,3,4,5,* and *100* in Eq.(1) are shown in Figs.7 - 18. The distance *r* is the function consisting of the length *l*.

$$\frac{d^2 r(l)}{dl^2} = -n \cdot r(l)^{2n-1} \quad n = 1,2,3,\cdots \tag{11}$$

The function *r(l)* is abbreviated as *r*. By multiplying the derivative *dr/dl,* Eq. (12) is obtained as follows:

$$\frac{dr}{dl}\frac{d^2 r}{dl^2} = -nr^{2n-1}\frac{dr}{dl} \quad n = 1,2,3,\cdots \tag{12}$$

By integrating both sides in Eq. (12), the following equation is obtained:

$$\frac{1}{2}\left(\frac{dr}{dl}\right)^2 = -\frac{1}{2}r^{2n} + C_1 \quad n = 1,2,3,\cdots \tag{13}$$

Using the initial condition in both Eq. (2) and Eq. (3), the constant C_1 is determined.

$$\frac{1}{2}\left(\frac{dr(0)}{dl}\right)^2 = -\frac{1}{2}r(0)^{2n} + C_1 \tag{14}$$

The following equation is obtained.

$$C_1 = \frac{1}{2} \tag{15}$$

By solving the derivative *dr/dl* in Eq. (13), the following equation is obtained.

$$\frac{dr}{dl} = \pm\sqrt{1 - r^{2n}} \tag{16}$$

In Fig.7, the arc length *l=0* indicates the distance *r=0.* As the variable *l* increases within the first quadrant ($0 \leqq l \leqq \pi/2$) in Fig.7, the variable *r* increases. It is natural that the differential *dr/dl* is defined as positive. Therefore, it is obtained as follows:

$$\frac{dr}{dl} = \sqrt{1 - r^{2n}} \tag{17}$$

In this section, notice that variables *r* and *l* only occur in the first quadrant. As described in section 5.2, with respect to the range of the variable *l*, it is necessary to decide the sign of the differential *dr/dl.* After separating the variables, Eq. (16) is integrated from *0* to *r* and is obtained as follows:

$$\int_0^r \frac{1}{\sqrt{1-t^{2n}}}dt = l \quad -1 \le r \le 1 \tag{18}$$

The inverse function of Eq. (18) is defined as follows:

$$arcsleaf_n(r) = \int_0^r \frac{1}{\sqrt{1-t^{2n}}}dt = l \tag{19}$$

The following equation is obtained.

$$r = sleaf_n(l) \tag{20}$$

In the case of *n=1*, the curve is shown in Fig. 7 and Fig. 8. The following equation is obtained.

$$sleaf_1(l) = \sin(l) \tag{21}$$

In the case of *n=1*, the arc length *l* is proportional to the radian angle.

$$l=\theta \tag{22}$$

Therefore, Eq.(20) is as follows:

$$sleaf_1(l) = \sin(\theta) \tag{23}$$

In the case of *n=2*, the curve is shown in Fig.9 and Fig.10 and the following equation is obtained.

$$sleaf_2(l) = sn(l,i) \tag{24}$$

The function *sn* represents Eq. (9). The variable *i* represents an imaginary number.

3.4 Relation between the geometry and the function: *sleaf$_n$(l)*

In this section, the relation between the geometry and the function *sleaf$_n$(l)* is described. The coordinate system of the function *sleaf$_n$(l)* is shown as polar coordinates.

$$x = r\cos(\theta) \tag{25}$$

$$y = r\sin(\theta) \tag{26}$$

The functions x and y consist of both the variables θ and r. Eq.(25) and Eq.(26) are differentiated with respect to the variable r to obtain the following equation.

$$\frac{dx}{dr} = \cos(\theta) - r\sin(\theta)\cdot\frac{d\theta}{dr} \tag{27}$$

$$\frac{dy}{dr} = \sin(\theta) + r\cos(\theta)\cdot\frac{d\theta}{dr} \tag{28}$$

In a small domain, approximation of the length Δl on the curve is shown as follows:

$$\Delta l = \sqrt{\Delta x^2 + \Delta y^2} = \sqrt{\left(\frac{\Delta x}{\Delta r}\right)^2 + \left(\frac{\Delta y}{\Delta r}\right)^2}\cdot\Delta r \tag{29}$$

If the variable Δl takes an infinitely small value, the following equation is obtained.

$$dl = \sqrt{\left(\frac{dx}{dr}\right)^2 + \left(\frac{dy}{dr}\right)^2}\cdot dr \tag{30}$$

By substituting Eq. (27) and (28) in Eq. (30), the following equation is obtained.

$$\begin{aligned} dl &= \sqrt{\left(\frac{dx}{dr}\right)^2 + \left(\frac{dy}{dr}\right)^2}\cdot dr \\ &= \sqrt{\left(\cos(\theta) - r\sin(\theta)\cdot\frac{d\theta}{dr}\right)^2 + \left(\sin(\theta) + r\cos(\theta)\cdot\frac{d\theta}{dr}\right)^2}\cdot dr \\ &= \sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2}\cdot dr \end{aligned} \tag{31}$$

By differentiating Eq. (10) with respect to the variable θ, the following equation is obtained.

$$nr^{n-1}\frac{dr}{d\theta} = n\cos(n\theta) \tag{32}$$

The above equation is as follows:

$$\frac{d\theta}{dr} = \frac{r^{n-1}}{\cos(n\theta)} \tag{33}$$

By substituting Eq. (33) in Eq. (31), the following equation is obtained.

$$\begin{aligned} dl &= \sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2}\cdot dr = \sqrt{1 + r^2\left(\frac{r^{n-1}}{\cos(n\theta)}\right)^2}\cdot dr \\ &= \sqrt{1 + \frac{r^{2n}}{\cos^2(n\theta)}}\cdot dr = \sqrt{1 + \frac{r^{2n}}{1 - \sin^2(n\theta)}}\cdot dr \\ &= \sqrt{1 + \frac{r^{2n}}{1 - r^{2n}}}\cdot dr = \frac{1}{\sqrt{1 - r^{2n}}}\cdot dr \end{aligned} \tag{34}$$

By integrating $\dfrac{1}{\sqrt{1 - r^{2n}}}$ from 0 to r, the following equation is obtained.

$$l = \int_0^r \frac{1}{\sqrt{1 - t^{2n}}}dt \tag{35}$$

The above equation is the same as the inverse function defined by Eq. (17). The following equation is obtained.

$$l = \int_0^r \frac{1}{\sqrt{1 - t^{2n}}}dt = arcsleaf_n(r) \tag{36}$$

The following equation is obtained.

$$r = sleaf_n(l) \tag{37}$$

By differentiating Eq. (35) with respect to the variable r, the following equation is obtained.

$$\frac{dl}{dr} = \frac{1}{\sqrt{1 - r^{2n}}} \tag{38}$$

The above equation is obtained as follows:

$$\left(\frac{dr}{dl}\right)^2 = 1 - r^{2n} \tag{39}$$

By differentiating the above equation with respect to the variable *l*, the following equation is obtained.

$$2\frac{dr}{dl}\frac{d^2 r}{dl^2} = -2nr^{2n-1}\frac{dr}{dl} \tag{40}$$

By reason of the condition $dr/dl \neq 0$, the following equation is obtained.

$$\frac{d^2 r}{dl^2} = -nr^{2n-1} \tag{41}$$

Using Eq. (36), the following equation is obtained.

$$\frac{d^2}{dl^2} sleaf_n(l) = -n \cdot \left(sleaf_n(l)\right)^{2n-1} \tag{42}$$

Therefore, Eqs. (1)-(3) can be described by the leaf function.

4. Numerical examination of leaf function

In the case of *n=1,2,3,4,5,* and *100* in Eq.(10) and (37), the graph is plotted. In this section, the variables *θ, x,* and *y* are only in the first quadrant. Therefore these variables are satisfied as follows:

$$0 \le \theta \le \frac{\pi}{2} \tag{43}$$

$$0 \le x \le 1 \tag{44}$$

$$0 \le y \le 1 \tag{45}$$

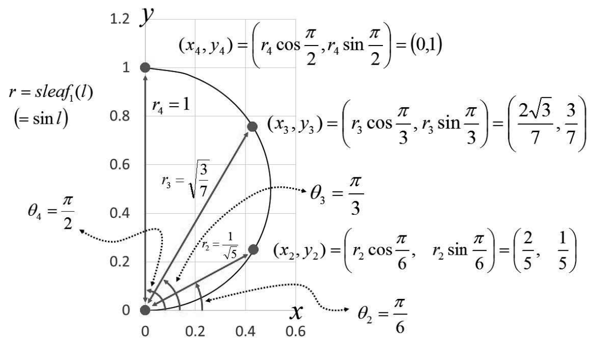
In the case of *n=1*, the function is as follows:

$$r = \sin(\theta) \tag{46}$$

Using Eq. (6), Eq. (25), and Eq. (26), the relation between the variables *r* and *θ* can be described by the relation between the variables *x* and *y*. It is obtained as follows:

$$x^2 + y^2 = y \tag{47}$$

These graphs are shown in Fig.7 and Fig.8.



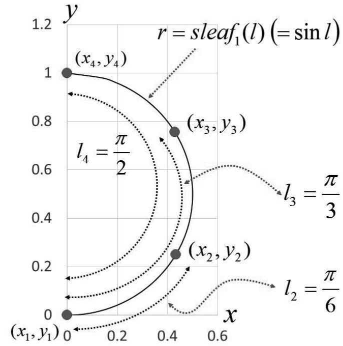


Fig.7 Leaf curve of *n=1 (0≦θ≦π/2)*
(Vertical and horizontal axes are set to *x* and *y*, respectively)

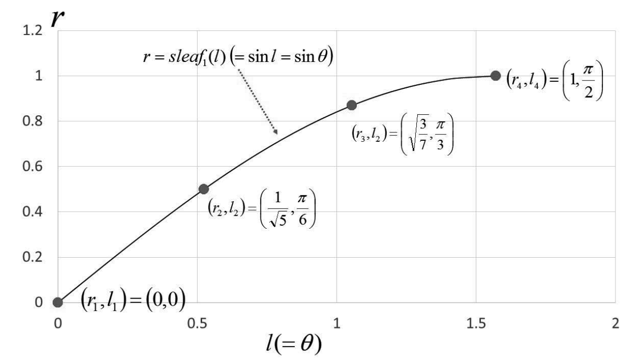


Fig.8 Leaf curve of *n=1 (0≦θ≦π/2)*
(Vertical and horizontal axes are set to *r* and *l*, respectively)

In the case of *n=2,* the function is as follows:

$$r^2 = \sin(2\theta) \tag{48}$$

Using Eq. (6), Eq. (25), and Eq. (26), the relation between the variables *r* and *θ* can be described by the relation between the variables *x* and *y*, which is obtained as follows:

$$\left(x^2 + y^2\right)^2 = 2xy \tag{49}$$

These graphs are shown in Fig.9 and Fig.10.

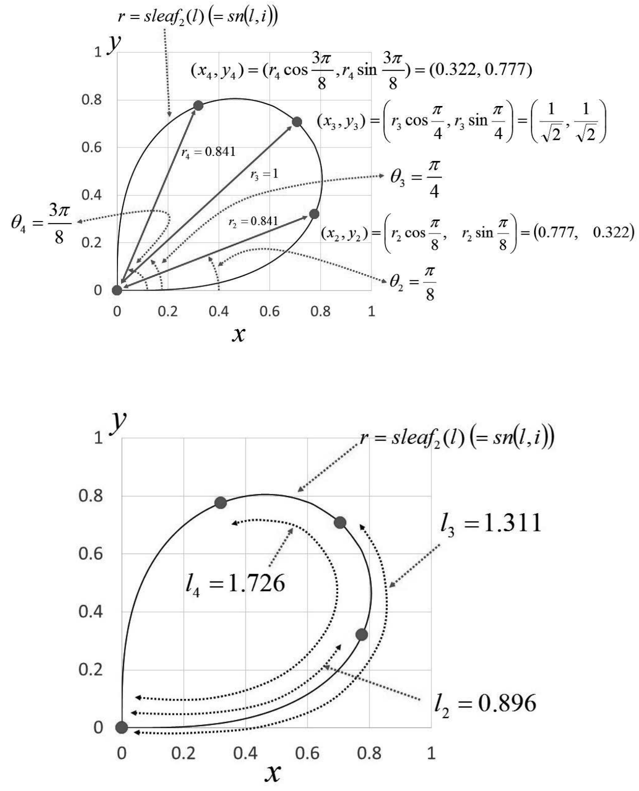


Fig.9 Leaf curve of *n=2 (0≦θ≦π/2)*
(Vertical and horizontal axes are set to *y* and *x*, respectively)

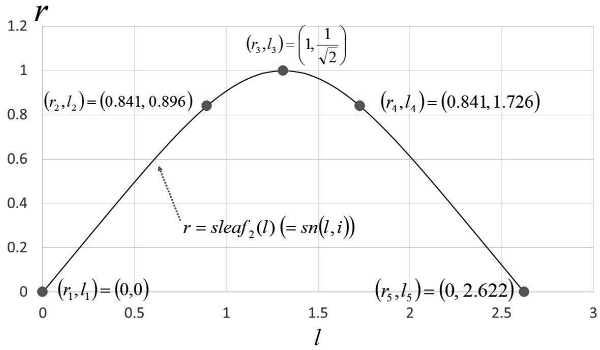


Fig.10 Leaf curve of *n=2 (0≦θ≦π/2)*
(Vertical and horizontal axes are set to *r* and *l*, respectively)

In the case of *n=3*, the function is as follows:

$$r^3 = \sin 3\theta \tag{50}$$

Using Eq. (6), Eq. (25), and Eq. (26), the relation between the variables *r* and *θ* can be described by the relation between the variables *x* and *y*, which is obtained as follows:

$$\left(x^2 + y^2\right)^3 = 3yx^2 - y^3 \tag{51}$$

These graphs are shown in Fig.11 and Fig.12.

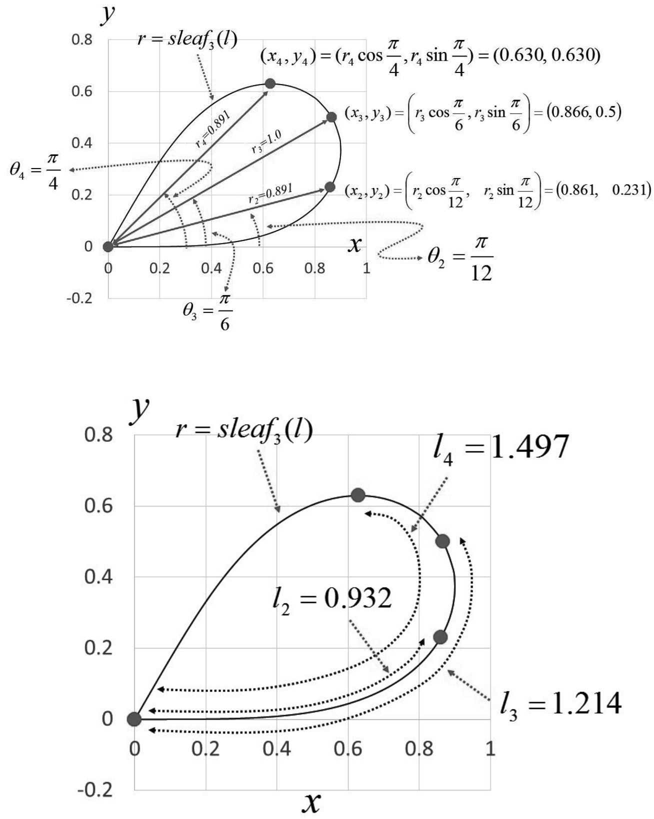


Fig.11 Leaf curve of *n=3 (0≦θ≦π/3)*
(Vertical and horizontal axes are set to *y* and *x*, respectively)

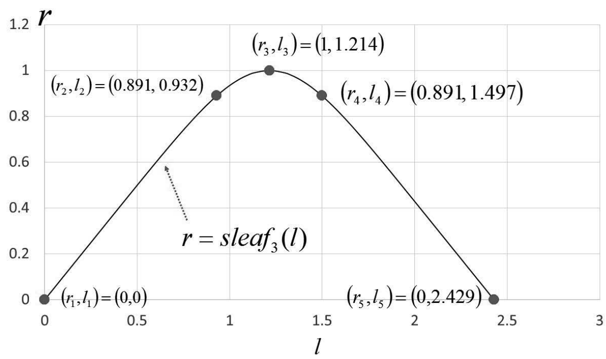


Fig.12 Leaf curve of *n=3 (0≦θ≦π/3)*
(Vertical and horizontal axes are set to *r* and *l*, respectively)

In the case of *n=4*, the function is as follows:

$$r^4 = \sin 4\theta \tag{52}$$

Using Eq. (6), Eq. (25), and Eq. (26), the relation between the variables *r* and *θ* can be described by the relation between the variables x and y, which is obtained as follows:

$$\left(x^2 + y^2\right)^4 = 4yx^3 - 4y^3x \tag{53}$$

These graphs are shown in Fig.13 and Fig.14.

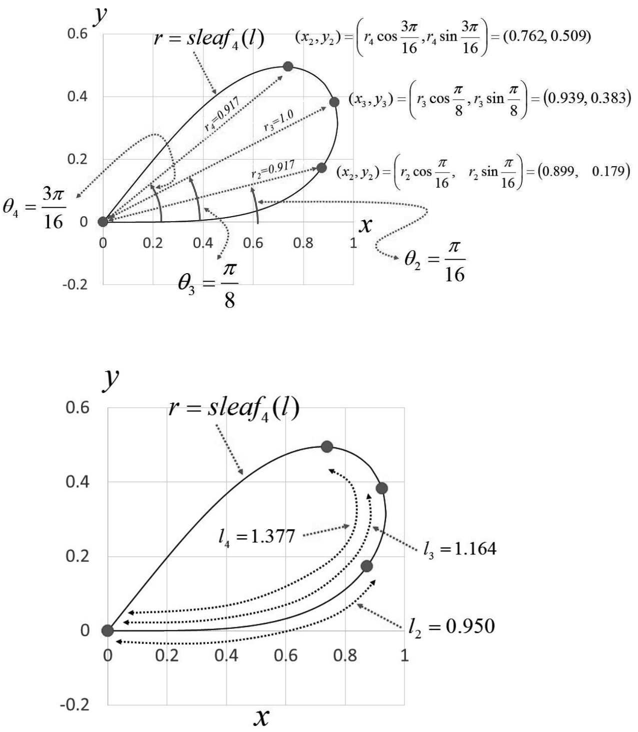


Fig.13 Leaf curve of $n=4$ $(0 \leqq \theta \leqq \pi/4)$ (Vertical and horizontal axes are set to y and x, respectively)

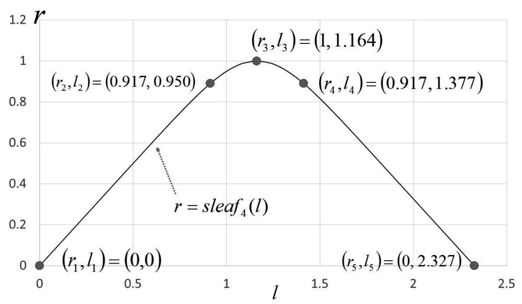


Fig.14 Leaf curve of $n=4$ $(0 \leqq \theta \leqq \pi/4)$
(Vertical and horizontal axes are set to r and l, respectively)

In the case of $n=5$, the function is as follows:

$$r^5 = \sin 5\theta \tag{54}$$

Using Eq. (6), Eq. (25), and Eq. (26), the relation between the variables r and θ can be described by the relation between the variables x and y, which is obtained as follows:

$$\left(x^2 + y^2\right)^5 = y^5 + 5yx^4 - 10y^3x^2 \tag{55}$$

These graphs are shown in Fig.15 and Fig.16.

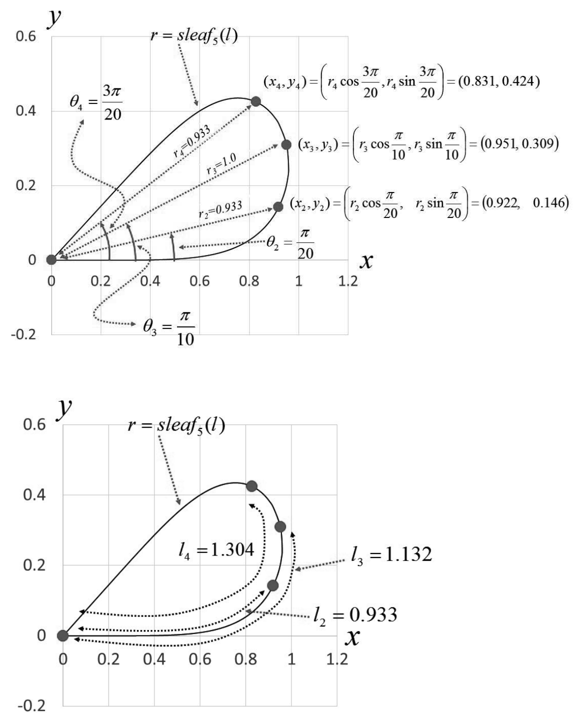


Fig.15 Leaf curve of $n=5$ $(0 \leqq \theta \leqq \pi/5)$
(Vertical and horizontal axes are set to y and x, respectively)

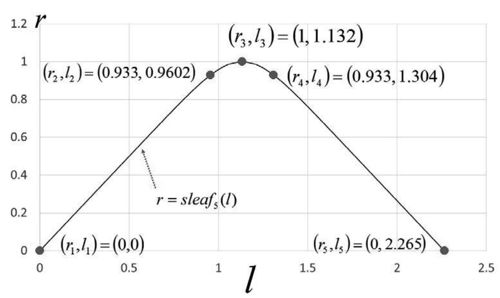


Fig.16 Leaf curve of $n=5$ $(0 \leqq \theta \leqq \pi/5)$
(Vertical and horizontal axes are set to r and l, respectively)

In the case of $n=100$, the function is as follows:

$$r^{100} = \sin(100\theta) \tag{56}$$

These graphs are shown in Fig.17 and Fig.18.

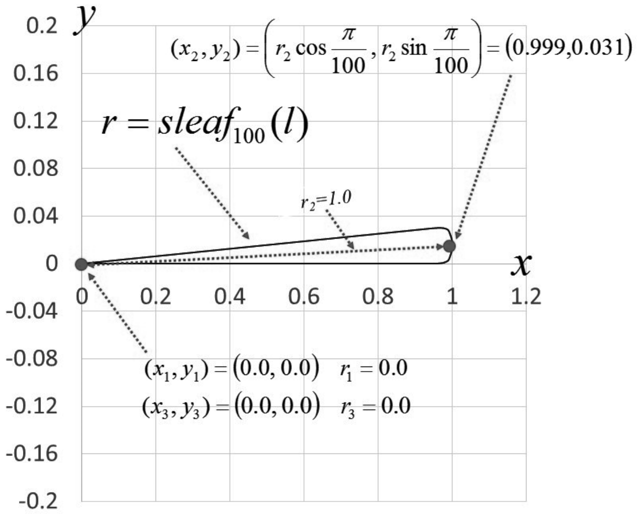


Fig.17 Leaf curve of *n=100 (0≦θ≦π/100)*
(Vertical and horizontal axes are set to *y* and *x*, respectively)

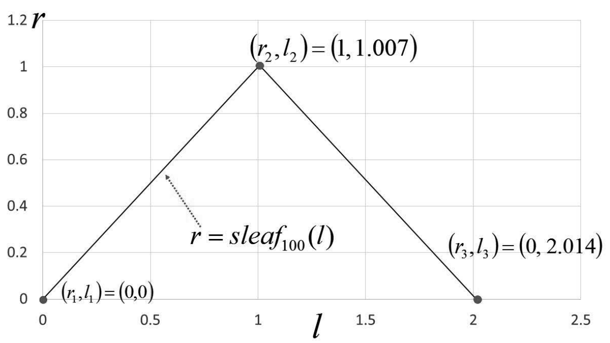


Fig.18 Leaf curve of *n=100 (0≦θ≦π/100)*
(Vertical and horizontal axes are set to *r* and *l*, respectively)

5. Re-examination of leaf function

5.1 Leaf curve ($x-y$ *plane*)

In earlier discussions, the leaf curve was described geometrically by assuming the variable *r* to have the range *r ≧0*. Therefore, various problems occur in the leaf function. In the case of an odd number *n* in Eq. (1), the inequality *sin(nθ)<0* in Eq. (10) exist for an arbitrary variable *θ*. The distance *r* has a negative value, geometrically and a negative distance cannot be described using a graph, geometrically. On the other hand, a negative *r* occurs in Eq. (10).

In the case of an even number *n*, the right side r^n in Eq.(10) consistently becomes positive, even if the left side *sin(nθ)* in Eq.(10) becomes negative. Therefore, for real numbers, Eq.(10) is not satisfied for an arbitrary variable *θ*. The function (*r –θ* function) is redefined as follows:

$$|r|^n = |\sin(n\theta)| \quad n = 1,2,3,\cdots \tag{57}$$

In the above equation, the parameter *n* is a natural number. The variable *r* includes both positive and negative numbers. By replacing *r* by |*r*|, and replacing *sin(nθ)* by | *sin(nθ)*|, the leaf curve can be related geometrically.

In the case of *n=1*, the leaf curve is shown in Fig.19. Compared to Fig.1, an additional leaf is added in the range (*π≦θ≦2π*), which occurs in the third and fourth quadrants. In Fig.19, the leaf (*r≧0*) in the first and second quadrants is defined as the positive leaf. The leaf (*r<0*) in the third and fourth quadrants is defined as the negative leaf. As shown in Fig.19, the leaf curve is defined as consisting of one positive and one negative leaf.

In the case of *n=2,3,4,5,* and *100*, the graphs on the *x-y* plane are shown in Fig.20-24. In this paper, the number of leaves is even, with positive and negative leaves arranged in alternating in order. The polar coordinates in Eq. (25)-(26) are redefined as follows:

$$x = |r|\cos\theta \tag{58}$$

$$y = |r|\sin\theta \tag{59}$$

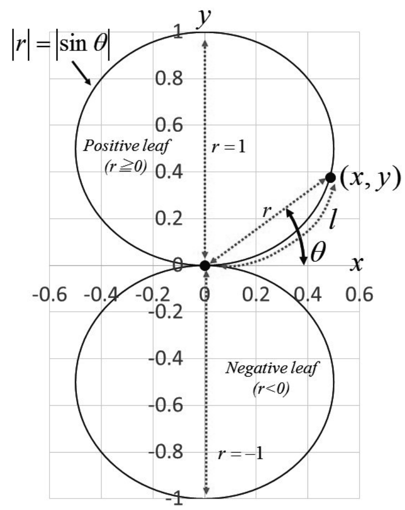


Fig.19 One positive - one negative leaf curve.

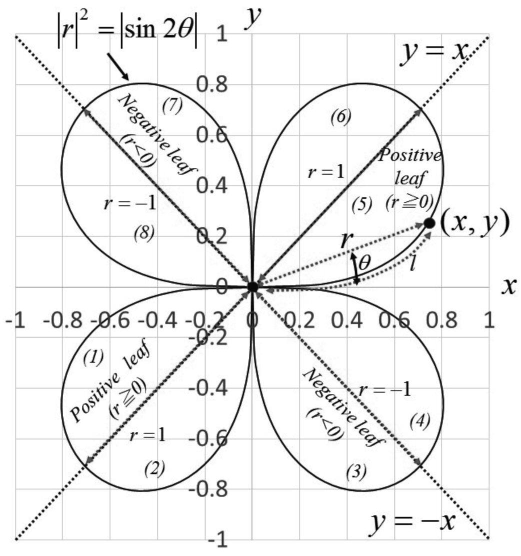


Fig. 20 Two positive - two negative leaf curve.

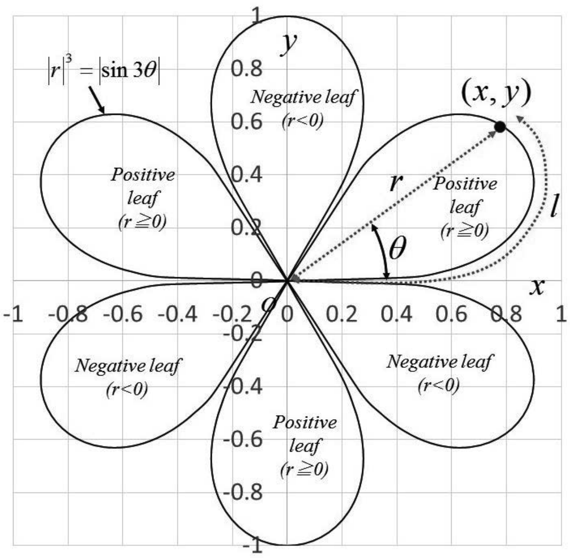


Fig. 21 Three positive - three negative leaf curve.

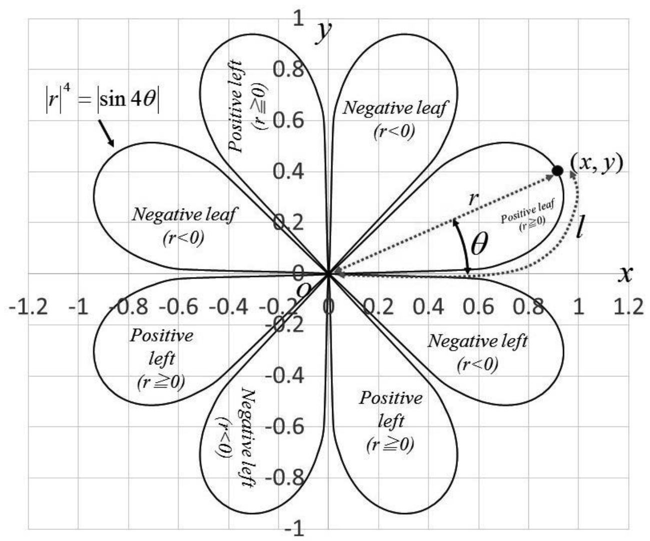


Fig. 22 Four positive - four negative leaf curve.

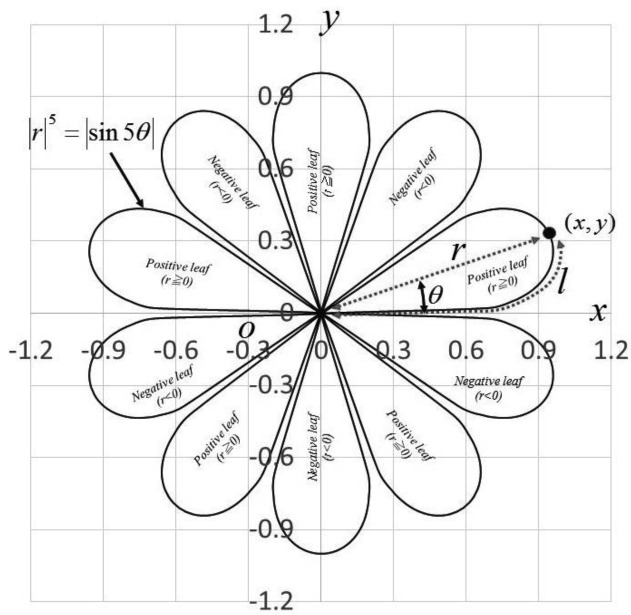


Fig. 23 Five positive - five negative leaf curve.

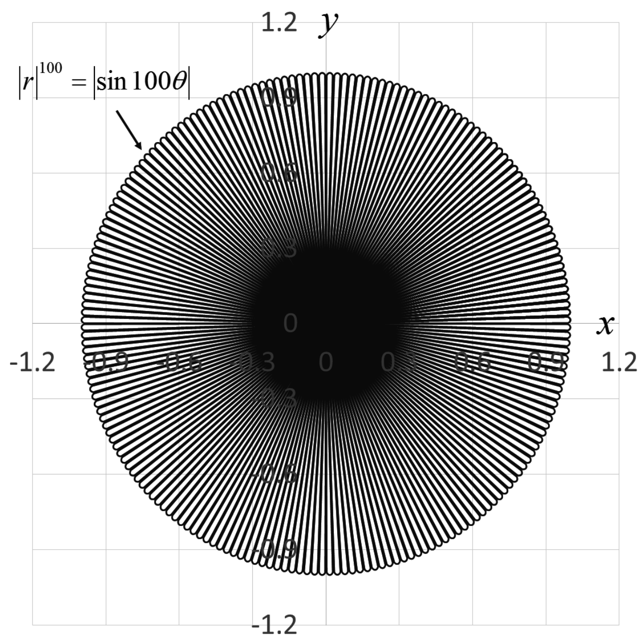


Fig. 24 Hundred positive - Hundred negative leaf curve.

5.2 Extended definition of leaf function

The constants $\pi_n/2$ are defined as follows:

$$l = \int_0^1 \frac{1}{\sqrt{1-t^{2n}}}dt = \frac{\pi_n}{2} \quad (n = 1,2,3,\cdots) \tag{60}$$

In the case of $n=1$, the constant π_1 represents the circular constant π. The constants π_n with respect to $n=1,2,3,4,5,$ and 100 are summarized in Table 1. Numerical values π_n are rounded off to five decimal places, and calculated with precision up to four digits.

Table 1 Values of constant π_n

n	π_n
1	π_1=3.142
2	π_2=2.622
3	π_3=2.429
4	π_4=2.327
5	π_5=2.265
100	π_{100}=2.014

As shown in Fig.1 - Fig.5, the constant π_n geometrically represents the circumference length of one leaf. The leaf function $sleaf_n(l)$ takes the constant $2\times\pi_n$ with respect to one period. In the angle θ, the counter-clockwise direction is defined as positive. As the angle θ increases from 0 to $\pi_n/2$, the distance increases from 0 to 1. Using Eq. (18), one input of the arc length l is calculated with respect to one output of variable r. The leaf function $sleaf_n(l)$ is defined as a multivalued function, with one input associated with multiple outputs.

First, the parameter $n=2$ in Eq. (35) is discussed. In the range $0\leqq\theta<\pi/4$ (domain (5) in Table 2 and Fig.20), the variable l is calculated as follows:

$$l=\int_0^r \frac{1}{\sqrt{1-t^4}}dt \quad (0\le r\le 1) \tag{61}$$

In the range $\pi/4\leqq\theta<\pi/2$ (domain (6) in Table 2 and Fig.20), using Eq. (16) with respect to r, the equation is obtained as follows:

$$\frac{dl}{dr}=\pm\frac{1}{\sqrt{1-r^{2n}}} \tag{62}$$

In the range of $\pi/4\leqq\theta<\pi/2$ in Eq. (10), the distance r varies from $r=1$ to $r=0$, with the variable r decreasing in the range. The sign of the variation dr is negative; thus, the sign of the above equation becomes negative.

$$\frac{dl}{dr}=-\frac{1}{\sqrt{1-r^4}} \quad \left(\frac{\pi}{4}\le\theta\le\frac{\pi}{2}\right) \tag{63}$$

In the range $\pi/4\leqq\theta<\pi/2$, the arc length is as follows:

$$\begin{aligned} l&=\int_0^1 \frac{1}{\sqrt{1-t^4}}dt+\int_1^r \frac{-1}{\sqrt{1-t^4}}dt \\ &=\frac{\pi_2}{2}+\int_r^1 \frac{1}{\sqrt{1-t^4}}dt \quad (0\le r\le 1) \end{aligned} \tag{64}$$

The constant π_2 is given in Table 1. In the range $\pi/2\leqq\theta<3\pi/4$, the domain in the x-y graph is defined as the negative leaf. The sign of the variable r becomes negative. The variable r is increased with respect to the negative direction, and the sign of the variation dr becomes negative. On the other hand, the sign of the variation dl becomes positive by increasing the variable l. Therefore, the sign of Eq. (62) becomes negative.

$$\frac{dl}{dr}=-\frac{1}{\sqrt{1-r^4}} \quad \left(\frac{\pi}{2}\le\theta\le\frac{3\pi}{4}\right) \tag{65}$$

The length l is obtained as follows:

$$\begin{aligned} l&=\int_0^1 \frac{1}{\sqrt{1-t^4}}dt+\int_1^0 \frac{-1}{\sqrt{1-t^4}}dt+\int_0^r \frac{-1}{\sqrt{1-t^4}}dt \\ &=\pi_2+\int_r^0 \frac{1}{\sqrt{1-t^4}}dt \quad (-1\le r\le 0) \end{aligned} \tag{66}$$

In the range $3\pi/4\leqq\theta<\pi$ (domain (8) in Table 2 and Fig.20), the domain in the x-y graph is defined as the negative leaf. The sign of the variable r becomes negative. The variable r starts at $r=-1$ and finally reaches $r=0$. The variation dr becomes positive. On the other hand, the length l increases. The sign of the variation dl becomes positive. The sign of the variation dl/dr becomes positive.

$$\frac{dl}{dr}=\frac{1}{\sqrt{1-r^4}} \quad \left(\frac{3}{4}\pi\le\theta\le\pi\right) \tag{67}$$

The length l is obtained as follows:

$$\begin{aligned} l&=\int_0^1 \frac{1}{\sqrt{1-t^4}}dt+\int_1^0 \frac{-1}{\sqrt{1-t^4}}dt+\int_0^{-1} \frac{-1}{\sqrt{1-t^4}}dt \\ &+\int_{-1}^r \frac{1}{\sqrt{1-t^4}}dt=\frac{3}{2}\pi_2+\int_{-1}^r \frac{1}{\sqrt{1-t^4}}dt \quad (-1\le r\le 0) \end{aligned} \tag{68}$$

In the negative case, the variable l should also be required to be defined. In this paper, the sign of the angle θ is defined as positive with respect to the counter-clockwise direction. Corresponding to the angle θ, the length l is defined as positive. On the other hand, the sign of the angle θ is defined as negative with respect to the clockwise direction. Corresponding to the angle θ, the length l is defined as negative.

In the range $-\pi/4\leqq\theta\leqq0$ (domain (4) in Table 2 and Fig.20), the domain in the x-y graph is defined as the negative leaf. The sign of the variable r becomes negative. The variable r starts at $r=0$ and finally reaches $r=-1$. The sign of the variation dr becomes negative. On the other hand, the length l increases with respect to the negative direction. The sign of the variation dl becomes negative. The sign of the variation dl/dr becomes positive.

$$\frac{dl}{dr} = \frac{1}{\sqrt{1-r^{2n}}} \quad \left(-\frac{1}{4}\pi \le \theta \le 0\right) \tag{69}$$

The length l is obtained as follows:

$$l = \int_0^r \frac{1}{\sqrt{1-t^4}} dt \quad (-1 \le r \le 0) \tag{70}$$

In the range $-\pi/2 \leqq \theta \leqq -\pi/4$ (domain (3) in Table 2 and Fig.20), the domain in the x-y graph is defined as the negative leaf. The sign of the variable r becomes negative. The variable r starts at $r=-1$ and finally reaches $r=0$. The sign of the variation dr becomes positive. On the other hand, the length l increases with respect to the negative direction. The sign of the variation dl becomes negative. The sign of the variation dl/dr becomes negative.

$$\frac{dl}{dr} = -\frac{1}{\sqrt{1-r^{2n}}} \quad \left(-\pi \le \theta \le -\frac{1}{2}\pi\right) \tag{71}$$

The length l is obtained as follows:

$$\begin{aligned} l &= \int_0^{-1} \frac{1}{\sqrt{1-t^4}} dt + \int_{-1}^r \frac{-1}{\sqrt{1-t^4}} dt \\ &= -\frac{\pi_2}{2} + \int_{-1}^r \frac{-1}{\sqrt{1-t^4}} dt \quad (-1 \le r \le 0) \end{aligned} \tag{72}$$

In the range $-3\pi/4 \leqq \theta \leqq -\pi/2$ (domain (2) in Table 2 and Fig.20), the domain in the x-y graph is defined as the positive leaf. The sign of the variable r becomes positive. The variable r starts at $r=0$ and finally reaches $r=1$. The sign of the variation dr becomes positive. On the other hand, the length l increases with respect to negative direction. The sign of the variation dl becomes negative. The sign of the variation dl/dr becomes negative.

$$\frac{dl}{dr} = -\frac{1}{\sqrt{1-r^{2n}}} \quad \left(-\frac{3}{2}\pi \le \theta \le -\pi\right) \tag{73}$$

The length l is obtained as follows:

$$\begin{aligned} l &= \int_0^{-1} \frac{1}{\sqrt{1-t^4}} dt + \int_{-1}^0 \frac{-1}{\sqrt{1-t^4}} dt + \int_0^r \frac{-1}{\sqrt{1-t^4}} dt \\ &= -\pi_2 + \int_0^r \frac{-1}{\sqrt{1-t^4}} dt \quad (0 \le r \le 1) \end{aligned} \tag{74}$$

In the range $-\pi \leqq \theta \leqq -3\pi/4$ (domain (1) in Table 2 and Fig.20), the domain in the x-y graph is defined as the positive leaf. The sign of the variable r becomes positive. The variable r starts at $r=1$ and finally reaches $r=0$. The sign of the variation dr becomes negative. On the other hand, the length l increases with respect to the negative direction. The sign of the variation dl becomes negative. The sign of the variation dl/dr becomes positive.

$$\frac{dl}{dr} = \frac{1}{\sqrt{1-r^{2n}}} \quad \left(-2\pi \le \theta \le -\frac{3}{2}\pi\right) \tag{75}$$

The length l is obtained as follows:

$$\begin{aligned} l &= \int_0^{-1} \frac{1}{\sqrt{1-t^4}} dt + \int_{-1}^0 \frac{-1}{\sqrt{1-t^4}} dt + \int_0^1 \frac{-1}{\sqrt{1-t^4}} dt \\ &+ \int_1^r \frac{1}{\sqrt{1-t^4}} dt = -\frac{3}{2}\pi_2 + \int_1^r \frac{1}{\sqrt{1-t^4}} dt \quad (0 \le r \le 1) \end{aligned} \tag{76}$$

In one period of both the positive and negative directions, the relation between the variables l and r is summarized in the case of $n=2$. For an arbitrary n, the same approach is applied. In the range $-2\pi_n \leqq l \leqq 2\pi_n$, the variables related to the function $sleaf_n(l)$ are summarized in Table 2 and Fig.25. With respect to the arbitrary n, the relation between the variables r and l is summarized in Table 3.

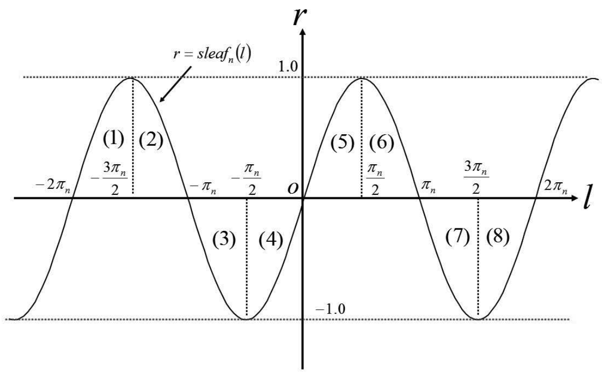


Fig. 25 Diagram of wave with respect to leaf function $sleaf_n(l)$ (In the figure, the numbers (1)-(8) represent the domain corresponding to Table 2 and Fig.20)

5.3 Waves of leaf function

Table 3 describes two types of graph. In the first type of graph, the vertical and horizontal axes are set to the variables r and l, respectively. In the second type of graph, the vertical and horizontal axes are set to the variables r and θ, respectively. The curves in both the x-y graph and the r-l graph are described as follows:

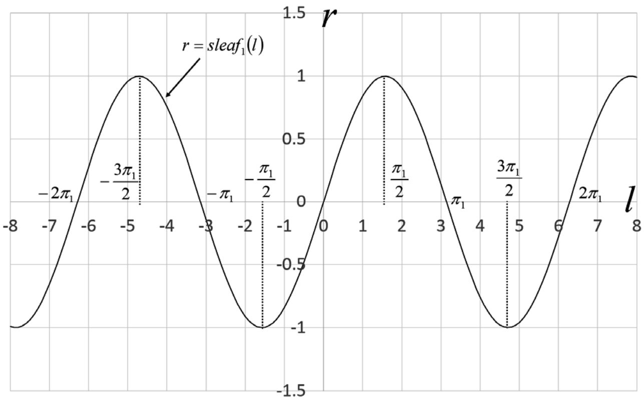


Fig. 26 Wave of leaf function $r=sleaf_1(l)$ $(=\sin(l))$
(1 period: $T=6.283(=2\pi_1)$)

Table 2 Relation between variables l and r for the leaf function $r=sleaf_n(l)$ with respect to one period in both the positive ($0 \leqq l \leqq 2\pi_n$) and negative directions ($-2\pi_n \leqq l \leqq 0$)

Domain	Range of angle θ	Range of length l	Length l	Range of variable r	Derivation dr/dl
(1)	$-2\pi\frac{1}{n} \le \theta < -\frac{3}{2}\pi\frac{1}{n}$	$-2\pi_n \le l < -\frac{3}{2}\pi_n$	$l=-\frac{3\pi_n}{2}-\int_r^1\frac{1}{\sqrt{1-t^{2n}}}dt$	$0 \leqq r \leqq 1$	$\frac{dr}{dl}=\sqrt{1-r^{2n}}$
(2)	$-\frac{3}{2}\pi\frac{1}{n} \le \theta < -\pi\frac{1}{n}$	$-\frac{3}{2}\pi_n \le l < -\pi_n$	$l=-\pi_n-\int_0^r\frac{1}{\sqrt{1-t^{2n}}}dt$	$0 \leqq r \leqq 1$	$\frac{dr}{dl}=-\sqrt{1-r^{2n}}$
(3)	$-\pi\frac{1}{n} \le \theta < -\frac{1}{2}\pi\frac{1}{n}$	$-\pi_n \le l < -\frac{1}{2}\pi_n$	$l=-\frac{\pi_n}{2}-\int_{-1}^r\frac{1}{\sqrt{1-t^{2n}}}dt$	$-1 \leqq r \leqq 0$	$\frac{dr}{dl}=-\sqrt{1-r^{2n}}$
(4)	$-\frac{1}{2}\pi\frac{1}{n} \le \theta < 0$	$-\frac{1}{2}\pi_n \le l < 0$	$l=\int_0^r\frac{1}{\sqrt{1-t^{2n}}}dt$	$-1 \leqq r \leqq 0$	$\frac{dr}{dl}=\sqrt{1-r^{2n}}$
(5)	$0 \le \theta < \frac{1}{2}\pi\frac{1}{n}$	$0 \le l < \frac{1}{2}\pi_n$	$l=\int_0^r\frac{1}{\sqrt{1-t^{2n}}}dt$	$0 \leqq r \leqq 1$	$\frac{dr}{dl}=\sqrt{1-r^{2n}}$
(6)	$\frac{1}{2}\pi\frac{1}{n} \le \theta < \pi\frac{1}{n}$	$\frac{1}{2}\pi_n \le l < \pi_n$	$l=\frac{\pi_n}{2}+\int_r^1\frac{1}{\sqrt{1-t^{2n}}}dt$	$0 \leqq r \leqq 1$	$\frac{dr}{dl}=-\sqrt{1-r^{2n}}$
(7)	$\pi\frac{1}{n} \le \theta < \frac{3}{2}\pi\frac{1}{n}$	$\pi_n \le l < \frac{3}{2}\pi_n$	$l=\pi_n+\int_r^0\frac{1}{\sqrt{1-t^{2n}}}dt$	$-1 \leqq r \leqq 0$	$\frac{dr}{dl}=-\sqrt{1-r^{2n}}$
(8)	$\frac{3}{2}\pi\frac{1}{n} \le \theta < 2\pi\frac{1}{n}$	$\frac{3}{2}\pi_n \le l < 2\pi_n$	$l=\frac{3\pi_n}{2}+\int_{-1}^r\frac{1}{\sqrt{1-t^{2n}}}dt$	$-1 \leqq r \leqq 0$	$\frac{dr}{dl}=\sqrt{1-r^{2n}}$

(Note) ・For domains (1)-(8), see Fig. 20 and Fig.25

・The derivation dr/dl represents the gradient of the function: $r=sleaf_n(l)$.

Table 3 Relation between the variables *r* and *l* of the leaf function $sleaf_n(l)$

Range of angle θ	Range of length l	Length l	Range of variable r	Derivation dr/dl
$(2m-2)\pi\frac{1}{n} \le \theta < \left(2m-\frac{3}{2}\right)\pi\frac{1}{n}$	$(2m-2)\pi_n \le l < \left(2m-\frac{3}{2}\right)\pi_n$	$l=(2m-2)\pi_n+\int_0^r \frac{1}{\sqrt{1-t^{2n}}}dt$	$0 \leqq r \leqq 1$	$\frac{dr}{dl}=\sqrt{1-r^{2n}}$
$\left(2m-\frac{3}{2}\right)\pi\frac{1}{n} \le \theta < (2m-1)\pi\frac{1}{n}$	$\left(2m-\frac{3}{2}\right)\pi_n \le l < (2m-1)\pi_n$	$l=\left(2m-\frac{3}{2}\right)\pi_n+\int_r^1 \frac{1}{\sqrt{1-t^{2n}}}dt$	$0 \leqq r \leqq 1$	$\frac{dr}{dl}=-\sqrt{1-r^{2n}}$
$(2m-1)\pi\frac{1}{n} \le \theta < \left(2m-\frac{1}{2}\right)\pi\frac{1}{n}$	$(2m-1)\pi_n \le l < \left(2m-\frac{1}{2}\right)\pi_n$	$l=(2m-1)\pi_n+\int_r^0 \frac{1}{\sqrt{1-t^{2n}}}dt$	$-1 \leqq r \leqq 0$	$\frac{dr}{dl}=-\sqrt{1-r^{2n}}$
$\left(2m-\frac{1}{2}\right)\pi\frac{1}{n} \le \theta < 2m\pi\frac{1}{n}$	$\left(2m-\frac{1}{2}\right)\pi_n \le l < 2m\pi_n$	$l=\left(2m-\frac{1}{2}\right)\pi_n+\int_{-1}^r \frac{1}{\sqrt{1-t^{2n}}}dt$	$-1 \leqq r \leqq 0$	$\frac{dr}{dl}=\sqrt{1-r^{2n}}$

(Note) The number *m* represents the integer ($m=0, \pm1, \pm2, \pm3, \pm4, \pm5, \cdots$)

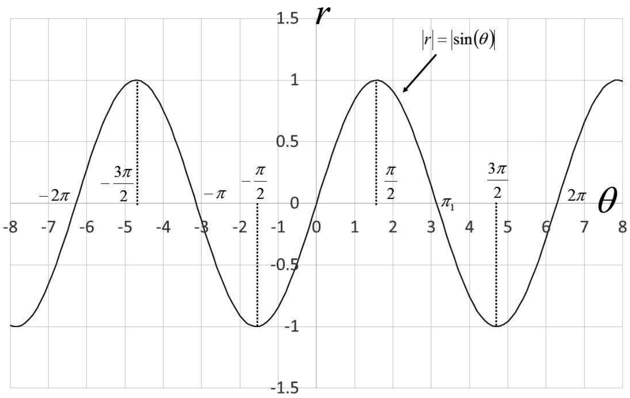


Fig. 27 Wave of leaf function $|r|=|sin(\theta)|$

(1 period: $T=\pi\times2$)

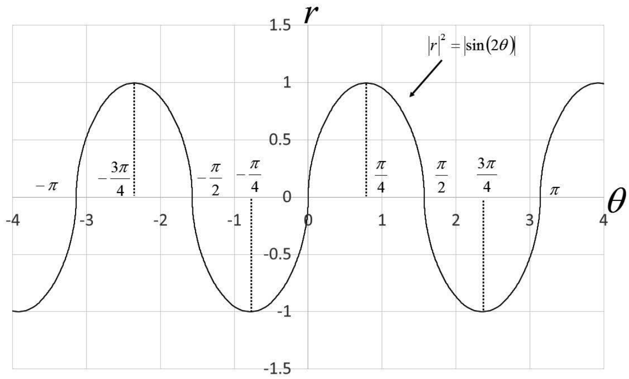


Fig. 29 Wave of leaf function $|r|^2=|sin(2\theta)|$

(1 period: $T=\pi/2\times2$)

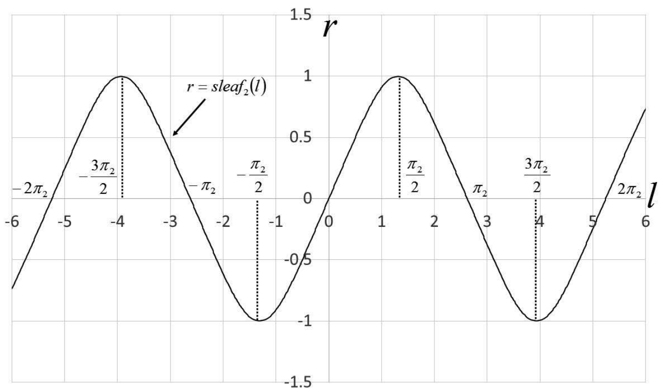


Fig. 28 Wave of leaf function $r=sleaf_2(l)$

(1 period: $T=5.244(=2\pi_2)$)

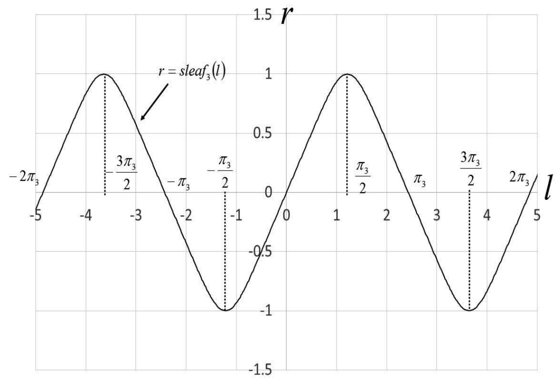


Fig. 30 Wave of leaf function $r=sleaf_3(l)$

(1 period: $T=4.857(=2\pi_3)$)

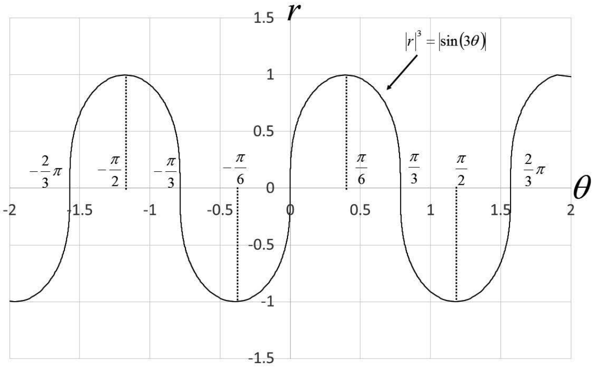


Fig. 31 Wave of leaf function $|r|^3=|sin(3\theta)|$

(1 period: $T=\pi/3\times2$)

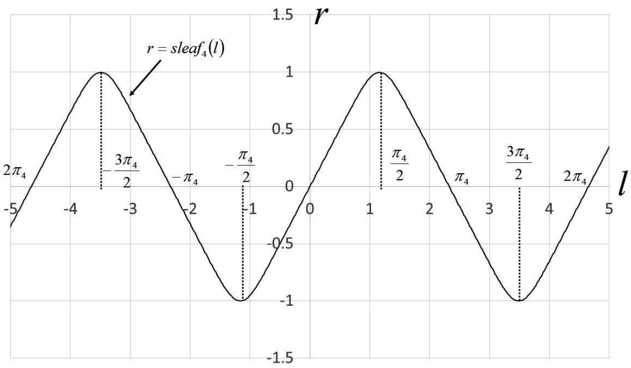


Fig. 32 Wave of leaf function $r=sleaf_4(l)$

(1 period: $T=4.654(=2\pi_4)$)

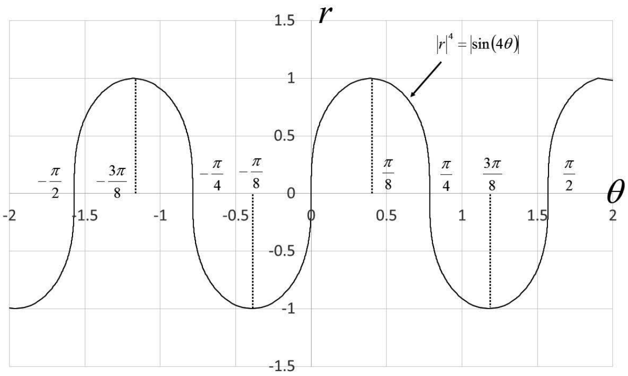


Fig. 33 Wave of leaf function $|r|^4=|sin(4\theta)|$

(1 period: $T=\pi/4\times2$)

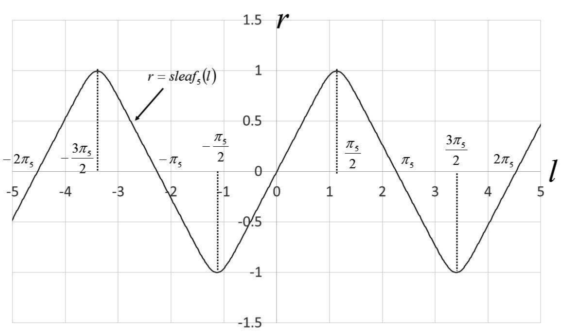


Fig. 34 Wave of leaf function $r=sleaf_5(l)$

(1 period: $T=4.529(=2\pi_5)$)

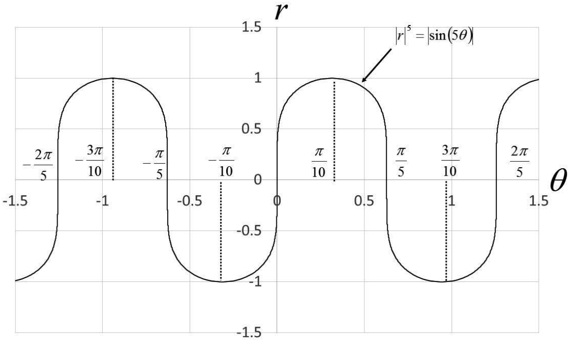


Fig. 35 Wave of leaf function $|r|^5=|sin(5\theta)|$

(1 period: $T=\pi/5\times2$)

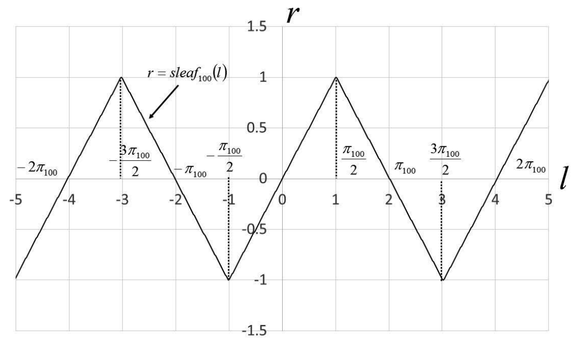


Fig. 36 Wave of leaf function $r=sleaf_{100}(l)$

(1 period: $T=4.028(=2\pi_{100})$)

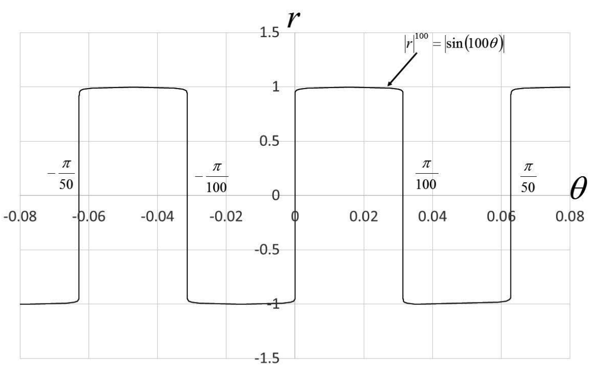


Fig. 37 Wave of leaf function $|r|^{100}=|sin(100\ \theta)|$

(1 period: $T=\pi/100\times2$)

6. Conclusion

The second derivative d^2r/dl^2 is equal to r^{2n-1}. This type of ODE has interesting features. Using numerical techniques, we can find that this ODE can produce a wave with periodicity. These waves are different from the waves obtained by trigonometric functions; therefore, a new function, the leaf function, is defined in this paper to describe these waves. The variable of the function consists of the variables *r* and *l*, which represent the distance between the origin and the point on the leaf curve and the length of the leaf curve, respectively. The relation between the variables and the geometry is also described. In the case of $n=1$ in a leaf function, the leaf curve is geometrically related to a circle and the leaf function is the trigonometric function $sin(\theta)$. In the case of $n=2$ in a leaf function, the function is the elliptical functions *sn(l,i)(i: imaginary number)*. As the parameter *n* increases, the waveform varies from a sine waveform to a saw-tooth waveform.

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