Special function: Leaf function $r=cleaf_n(l)$ (Second report)

Kazunori Shinohara*

Summary

In the previous report, the special function: leaf function $r=sleaf_n(l)$ was presented. The distance $|r|^n$ between the origin and the point on the leaf curve is equal to $sin(n\theta)$ (θ:angle, n:natural number). Using the equation $|r|^n=|sin(n\theta)|$, the shape of the leaf is described on the x-y plane. In this paper, the special function: leaf function $cleaf_n(l)$ is presented. The relation between the function $cleaf_n(l)$ and the function $sleaf_n(l)$ is described.

Keywords : Leaf function, Leaf curve, Jacobi elliptic functions, Elliptic integrals, Lemniscate, Ordinary differential equation, Addition theorem, Square root of polynomial

1. Introduction

In this paper, variables are always real numbers. Complex numbers are not considered. We follow the ordinary differential equation (ODE):

$$\frac{d^2r(l)}{dl^2}=-n\cdot r(l)^{2n-1} \quad (1)$$

$$r(0)=1 \quad (2)$$

$$\frac{dr(0)}{dl}=0 \quad (3)$$

The variable $r(l)$ represents a function with respect to the variable l. Equations (2) and (3) represent the initial conditions of an ODE. The number n represents a natural number $(n=1,2,3,\cdots)$.

In the paper, a leaf function $cleaf_n(l)$ satisfying Eqs. (1)-(3) is presented. The relation between the leaf function and its geometry is described through numerical results by substituting $n=1,2,3,4,5,$ and 100 in Eq. (1).

2. Symbols

The symbols used in this paper are as follows:

n: Natural number ($n=1,2,3,\cdots$) In the paper, it is named as basis.

r: Distance between the origin and the point on the curve

$$r=\sqrt{x^2+y^2}\geq 0 \quad (4)$$

As described below, the negative variable r has to be defined in Eq. (1).

θ: This variable represents the angle. In this paper, the unit is radian. Counter-clockwise is positive. Clockwise is negative.

l: Arc length on a leaf curve

Numerical values are rounded off to five decimal places, and calculated with a precision of up to four digits.

*Department of Integrated Mechanical Engineering, Daido University
Address: 10-3 Takiharu-cho, Minami-ku, Nagoya, JAPAN
E-mail: shinohara@06.alumni.u-tokyo.ac.jp

3. Leaf function

3.1 Elliptic function [1]

The inverse Jacobi elliptic function *arccd* is defined as follows [1]:

$$l = arccd(r,k) = \int_r^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} \qquad (5)$$

$$-1 \le r \le 1$$

where parameter k is the modulus of the elliptic integral. The sign t represents a parameter. Therefore, Eq. (5) is as follows:

$$r = cd(l,k) \qquad (6)$$

3.2 Leaf curve (x - y plane)

In the first report, as geometrical features of the leaf function: $sleaf_n(l)$, the leaf curve is defined as follows:

$$r^n = \sin(n\theta) \quad n = 1,2,3,\cdots \quad (r \ge 0) \qquad (7)$$

When the curve is described by a graph consisting of two axes (the x-axis (the horizontal axis) and the y-axis (the vertical axis)), the shape of the curve is similar to the shape of the leaf. Therefore, the curve is defined as the leaf curve. As a pair of Eq. (7), the leaf curve is defined as follows:

$$r^n = \cos(n\theta) \quad n = 1,2,3,\cdots \quad (r \ge 0) \qquad (8)$$

In the case of $n=1$, the graph of the equation $r=\cos(\theta)$ is shown in Fig. 1. When the angle θ increases, the point $(x,y)=(1,0)$ is close to the origin along the circular arc. As shown in Fig.1, the curve is described on a graph consisting of two axes, the x-axis (the horizontal axis) and the y-axis (the vertical axis). The curve in Fig.1 represents the one right curve. In the case of $n=2$, the leaf curve is shown in Fig. 2. The leaf curve represents the lemniscate curve. In the case of $n=3,4,5,$ and 100, these curves are shown in Fig. 2 - Fig. 6. These curves are defined as the two positive curves, the three positive curves, the four positive curves, the five positive curves, and the hundred positive curves. When the number n is increased in Eq. (1), the number of leaves in the figures increases.

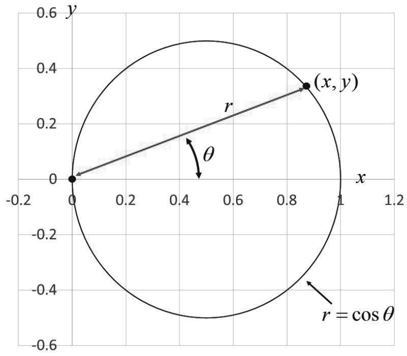

Fig. 1 One positive leaf curve (Circle of center (0.5, 0))

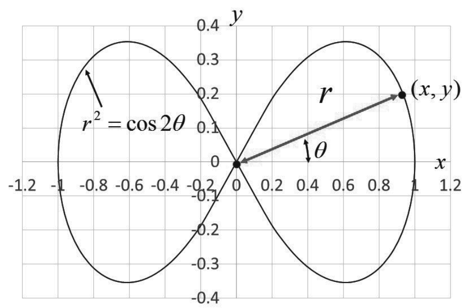

Fig. 2 Two positive leaf curve (lemniscate)

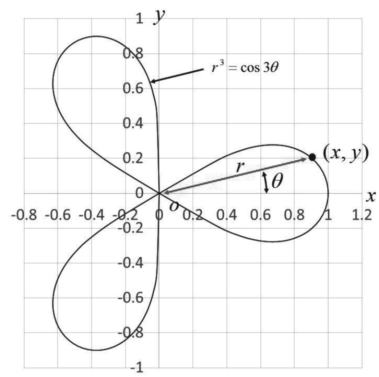

Fig. 3 Three positive leaf curve

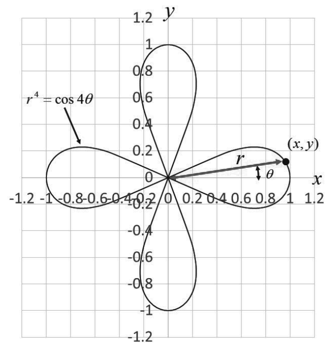

Fig. 4 Four positive leaf curve

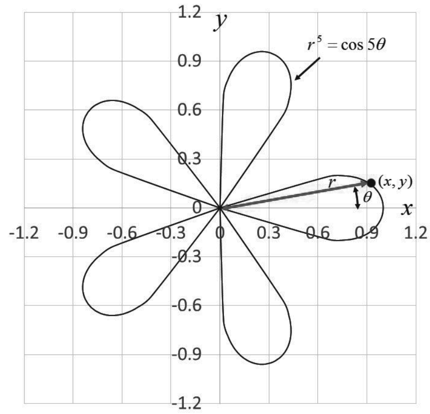

Fig. 5 Five positive leaf curve

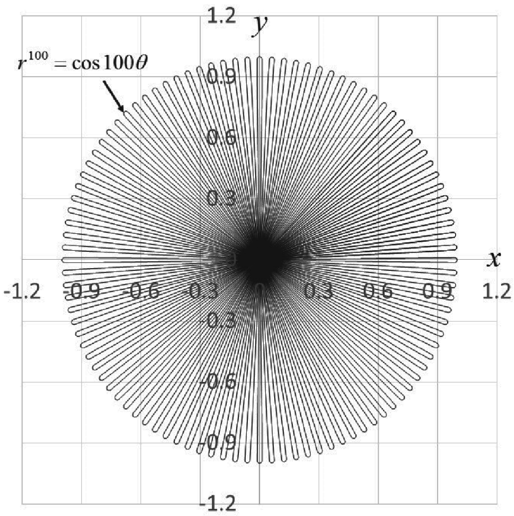

Fig. 6 Hundred positive leaf curve

3.3 Leaf function（*r-l* plane）（in first quadrant）

In this section, we discuss the ODE in Eq. (1). The parameter n represents a natural number. The variable l represents the length between the origin and the point on the leaf curve.

$$\frac{d^2 r(l)}{dl^2} = -n \cdot r(l)^{2n-1} \quad n = 1,2,3,\cdots \tag{9}$$

The function $r(l)$ is abbreviated as r. By multiplying the derivative dr/dl, the following equation is obtained:

$$\frac{dr}{dl}\frac{d^2 r}{dl^2} = -nr^{2n-1}\frac{dr}{dl} \quad n = 1,2,3,\cdots \tag{10}$$

By integrating the both sides of the above equation, the following equation is obtained:

$$\frac{1}{2}\left(\frac{dr}{dl}\right)^2 = -\frac{1}{2}r^{2n} + C_1 \quad n = 1,2,3,\cdots \tag{11}$$

Using the initial conditions in both Eq. (2) and Eq. (3), the constant C_1 is determined.

$$\frac{1}{2}\left(\frac{dr(0)}{dl}\right)^2 = -\frac{1}{2}r(0)^{2n} + C_1 \tag{12}$$

The following equation is obtained.

$$C_1 = \frac{1}{2} \tag{13}$$

By solving the derivative dr/dl in Eq. (11), the following equation is obtained.

$$\frac{dr}{dl} = \pm\sqrt{1 - r^{2n}} \tag{14}$$

In the leaf function: $sleaf_n(l)$, within the length range: $0 \leqq l \leqq \pi_n/2$, the above equation (the derivative dr/dl) is defined as the positive sign. In the leaf function: $cleaf_n(l)$, the above equation (the derivative dr/dl) is defined as the negative sign. For example, as shown in Fig.14, the variable $l=0$ becomes the variable $r=1$. As the length l increases, the variable r decreases in Fig.14. Therefore, the derivative dr/dl takes the negative as follows:

$$\frac{dr}{dl} = -\sqrt{1 - r^{2n}} \tag{15}$$

After separating the variables, Eq.(15) is integrated from 1 to r as follows:

$$\int_1^r \frac{-1}{\sqrt{1 - t^{2n}}} dt = l \tag{16}$$

The inverse function of Eq. (16) is defined as follows:

$$arccleaf_n(r) = \int_r^1 \frac{1}{\sqrt{1 - t^{2n}}} dt = l \tag{17}$$

The following equation is obtained.

$$r = cleaf_n(l) \tag{18}$$

In the case of $n=1$, the following equation is obtained.

$$cleaf_1(l)=\cos(l) \tag{19}$$

In the case of *n=1*, the angle *θ* is proportional to the arc length *l*.

$$l=\theta \tag{20}$$

Therefore, the equation can be described as follows:

$$cleaf_1(l)=\cos(\theta) \tag{21}$$

In the case of *n=2*, the following relation is obtained:

$$cleaf_2(l)=cd(l,i) \tag{22}$$

The elliptical function *cd* represents Eq. (6). The symbol *i* represents an imaginary number.

3.4 Relation between the geometry and the function: $cleaf_n(l)$

In this section, the relation between the geometry and the function $cleaf_n(l)$ is described. The coordinate system of the function $cleaf_n(l)$ is shown as polar coordinates.

$$x=r\cos\theta \tag{23}$$

$$y=r\sin\theta \tag{24}$$

The functions *x* and *y* contain both the variables *θ* and *r*. Eq. (23) and Eq. (24) are differentiated with respect to the variable *r*. The following equation is obtained.

$$\frac{dx}{dr}=\cos\theta-r\sin\theta\cdot\frac{d\theta}{dr} \tag{25}$$

$$\frac{dy}{dr}=\sin\theta+r\cos\theta\cdot\frac{d\theta}{dr} \tag{26}$$

In a small domain, the approximation of the length *Δl* on the curve is shown as follows:

$$\Delta l=\sqrt{\Delta x^2+\Delta y^2}=\sqrt{\left(\frac{\Delta x}{\Delta r}\right)^2+\left(\frac{\Delta y}{\Delta r}\right)^2}\cdot\Delta r \tag{27}$$

If the variable *Δl* takes an infinitely small value, the following equation is obtained.

$$dl=\sqrt{\left(\frac{dx}{dr}\right)^2+\left(\frac{dy}{dr}\right)^2}\cdot dr \tag{28}$$

By substituting Eq. (25) and (26) in the above equation, the following equation is obtained.

$$\begin{aligned} dl&=\sqrt{\left(\frac{dx}{dr}\right)^2+\left(\frac{dy}{dr}\right)^2}\cdot dr\\ &=\sqrt{\left(\cos\theta-r\sin\theta\cdot\frac{d\theta}{dr}\right)^2+\left(\sin\theta+r\cos\theta\cdot\frac{d\theta}{dr}\right)^2}\cdot dr\\ &=\sqrt{1+r^2\left(\frac{d\theta}{dr}\right)^2}\cdot dr \end{aligned} \tag{29}$$

By differentiating Eq. (8) with respect to the variable *θ*, the following equation is obtained.

$$nr^{n-1}\frac{dr}{d\theta}=-n\sin n\theta \tag{30}$$

The following equation is obtained.

$$\frac{d\theta}{dr}=-\frac{r^{n-1}}{\sin n\theta} \tag{31}$$

By substituting the above equation in Eq. (29), the following equation is obtained.

$$\begin{aligned} dl&=\sqrt{1+r^2\left(\frac{d\theta}{dr}\right)^2}\cdot dr=\sqrt{1+r^2\left(-\frac{r^{n-1}}{\sin(n\theta)}\right)^2}\cdot dr\\ &=\sqrt{1+\frac{r^{2n}}{(\sin(n\theta))^2}}\cdot dr=\sqrt{1+\frac{r^{2n}}{1-(\cos(n\theta))^2}}\cdot dr\\ &=\sqrt{1+\frac{r^{2n}}{1-r^{2n}}}\cdot dr=\frac{1}{\sqrt{1-r^{2n}}}\cdot dr \end{aligned} \tag{32}$$

By integrating the above equation from *1* to *r*, the following equation is obtained.

$$l=\int_1^r\frac{-1}{\sqrt{1-t^{2n}}}dt \quad (0\le r\le 1) \tag{33}$$

The above equation is the same as the inverse function defined by Eq. (17). The following equation is obtained.

$$l = \int_r^1 \frac{1}{\sqrt{1-t^{2n}}} dt = arccleaf_n(r) \tag{34}$$

The following equation is obtained.

$$r = cleaf_n(l) \tag{35}$$

By differentiating Eq. (34) with respect to the variable r, the following equation is obtained.

$$\frac{dl}{dr} = -\frac{1}{\sqrt{1-r^{2n}}} \tag{36}$$

The above equation is obtained as follows:

$$\left(\frac{dr}{dl}\right)^2 = 1 - r^{2n} \tag{37}$$

By differentiating the above equation with respect to the variable l, the following equation is obtained.

$$2\frac{dr}{dl}\frac{d^2r}{dl^2} = -2nr^{2n-1}\frac{dr}{dl} \tag{38}$$

By reason of the condition $dr/dl \neq 0$, the following equation is obtained.

$$\frac{d^2r}{dl^2} = -nr^{2n-1} \tag{39}$$

Using Eq. (35), the following equation is obtained.

$$\frac{d^2}{dl^2} cleaf_n(l) = -n \cdot (cleaf_n(l))^{2n-1} \tag{40}$$

Therefore, equations (1)-(3) can be described by the leaf function: $cleaf_n(l)$.

4. Numerical examination of leaf function

4.1 Leaf curve

In the previous section, we discussed the range: $r \geqq 0$. The leaf function: $r=cleaf_n(l)$ takes the range $r<0$ (the reason for this is provided in the first report). The geometry and the leaf curve: $cleaf_n(l)$ are related by redefining the leaf function r consisting of the variable θ as follows:

$$|r|^n = |\cos n\theta| \quad n = 1,2,3,\cdots \tag{41}$$

Using the above equation of $n=1,2,3,4,5,\cdots,100,$ the leaf curve is shown in Fig. 7 – 12.

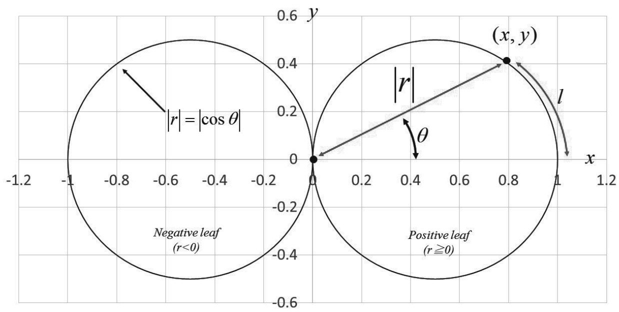

Fig. 7 One positive - one negative leaf curve.

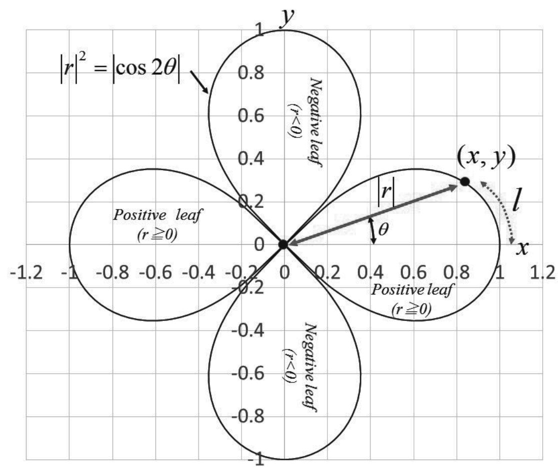

Fig. 8 Two positive - two negative leaf curve.

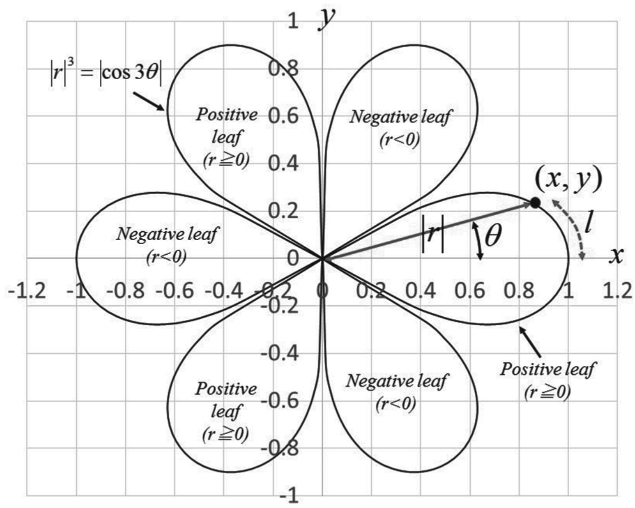

Fig. 9 Three positive - three negative leaf curve.

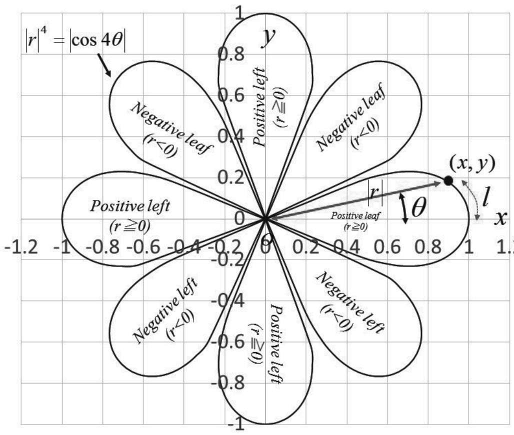

Fig. 10 Four positive - four negative leaf curve.

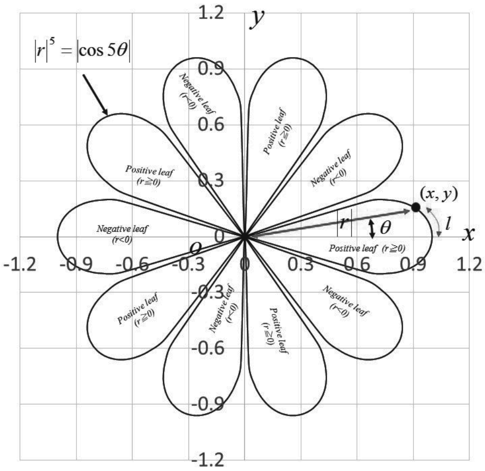

Fig. 11 Five positive - five negative leaf curve.

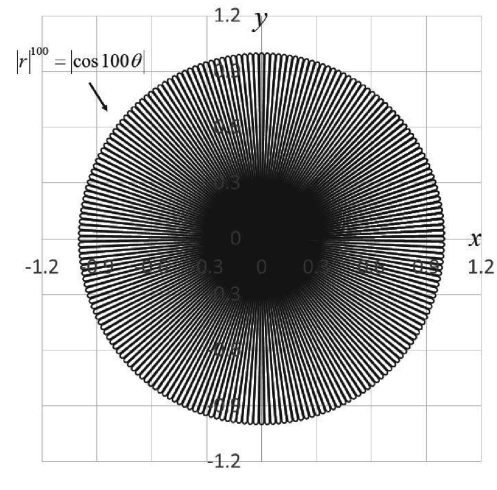

Fig. 12 Hundred positive - Hundred negative leaf curve.

4.2 Extended definition of leaf function

The constants $\pi_n/2$ are defined as follows:

$$\frac{\pi_n}{2} = \int_0^1 \frac{1}{\sqrt{1-t^{2n}}} dt \quad (n = 1,2,3,\cdots) \tag{42}$$

In the case of $n=1$, the constant π_1 represents the circular constant π. The constants π_n with respect to $n=1,2,3,4,5,$ and 100 are summarized in Table 1. The numerical values π_n are rounded off to five decimal places, and calculated with a precision of up to four digits.

Table 1 Values of constant π_n

n	π_n
1	π_1=3.142
2	π_2=2.622
3	π_3=2.429
4	π_4=2.327
5	π_5=2.265
100	π_{100}=2.014

The leaf function $cleaf_n(l)$ takes the constant $2\times\pi_n$ with respect to one period. For the angle θ, the counter-clockwise direction is defined as positive. As the angle θ increases from 0 to $\pi_n/2$, the distance decreases from 1 to 0. Using Eq. (33), one input of the arc length l is calculated with respect to one output of variable r. The leaf function $cleaf_n(l)$ is defined as a multivalued function, with one input associated with multiple outputs. First, we discuss the parameter $n=2$ in Eq. (34). In the range $0 \leqq \theta < \pi/4$ (domain (5) in Table 2 and Fig. 13), the variable l is calculated as follows:

$$l = \int_r^1 \frac{1}{\sqrt{1-t^4}} dt \quad (0 \le r \le 1) \tag{43}$$

In the range $\pi/4 \leqq \theta < \pi/2$ (domain (6) in Table 2 and Fig. 13), using Eq. (14) with respect to r, the equation is obtained as follows:

$$\frac{dl}{dr} = \pm \frac{1}{\sqrt{1-r^{2n}}} \tag{44}$$

In the range $\pi/4 \leqq \theta < \pi/2$, the variable r becomes $r<0$. The sign of the variation dr becomes negative as θ becomes increasingly negative. On the other hand, the length l increases in the positive direction. The sign of the variation dl becomes positive. Therefore, the sign of Eq. (44) becomes negative.

$$\frac{dl}{dr} = -\frac{1}{\sqrt{1-r^4}} \quad \left(\frac{\pi}{4} \le \theta \le \frac{\pi}{2}\right) \tag{45}$$

In the range $\pi/4 \leqq \theta < \pi/2$, the arc length l is as follows:

$$l = \int_0^1 \frac{1}{\sqrt{1-t^4}}dt + \int_0^r \frac{-1}{\sqrt{1-t^4}}dt = \frac{\pi_2}{2} + \int_r^0 \frac{1}{\sqrt{1-t^4}}dt \quad (-1 \le r \le 0) \tag{46}$$

The constant π_2 is given in Table 1. In the range $\pi/2 \leqq \theta < 3\pi/4$, the domain in the *x-y* graph is defined as a negative leaf. The sign of the variable *r* becomes negative. The variable *r* varies from $r=-1$ to $r=0$. The sign of the variation *dr* becomes positive. On the other hand, the length *l* increases. The sign of the variation *dl* becomes positive. The sign of the variation *dl/dr* becomes positive.

$$\frac{dl}{dr} = \frac{1}{\sqrt{1-r^4}} \quad \left(\frac{\pi}{4} \le \theta \le \frac{\pi}{2}\right) \tag{47}$$

The length *l* is obtained as follows:

$$l = \int_0^1 \frac{1}{\sqrt{1-t^4}}dt + \int_0^{-1} \frac{-1}{\sqrt{1-t^4}}dt + \int_{-1}^r \frac{1}{\sqrt{1-t^4}}dt = \pi_2 + \int_{-1}^r \frac{1}{\sqrt{1-t^4}}dt \quad (-1 \le r \le 0) \tag{48}$$

In the range $3\pi/4 \leqq \theta < \pi$, the domain in the *x-y* graph is defined as the positive leaf. The sign of the variable *r* becomes positive. The variable *r* varies from $r=-1$ to $r=0$. The sign of the variation *dr* becomes positive. On the other hand, the length *l* increases. The sign of the variation *dl* becomes positive. The sign of the variation *dl/dr* becomes positive.

$$\frac{dl}{dr} = \frac{1}{\sqrt{1-r^4}} \quad \left(\frac{3\pi}{4} \le \theta \le \pi\right) \tag{49}$$

The length *l* is obtained as follows:

$$l = \int_0^1 \frac{1}{\sqrt{1-t^4}}dt + \int_1^0 \frac{-1}{\sqrt{1-t^4}}dt + \int_{-1}^0 \frac{1}{\sqrt{1-t^4}}dt + \int_0^r \frac{1}{\sqrt{1-t^4}}dt = \frac{3}{2}\pi_2 + \int_0^r \frac{1}{\sqrt{1-t^4}}dt \quad (0 \le r \le 1) \tag{50}$$

In one period of both the positive and negative direction, the relation between variable *l* and *r* is summarized in the case of $n=2$. For an arbitrary *n*, the same approach is applied. In the range of $-2\pi_n \leqq l \leqq 2\pi_n$, the variable related to the function $cleaf_n(l)$ is summarized in Table 2 and Fig. 13. With respect to the arbitrary *n*, the relation between variable *l* and *r* is summarized in Table 3.

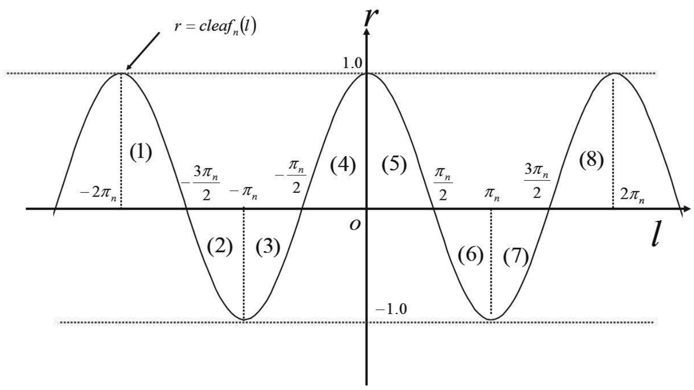

Fig. 13 Diagram of wave with respect to leaf function: $cleaf_n(l)$ (In the figure, the numbers (1) - (8) represent the domain corresponding to Table 2)

4.3 Waves of Leaf function

Two types of graph are shown in Figs. 14-25. In the first type of graph, the vertical and horizontal axes are set to variable *r* and *l*, respectively. In the second type of graph, the vertical and horizontal axes are set to variable *r* and *θ*, respectively. The curves of both the *x-y* graph and the *r-l* graph are described as follows:

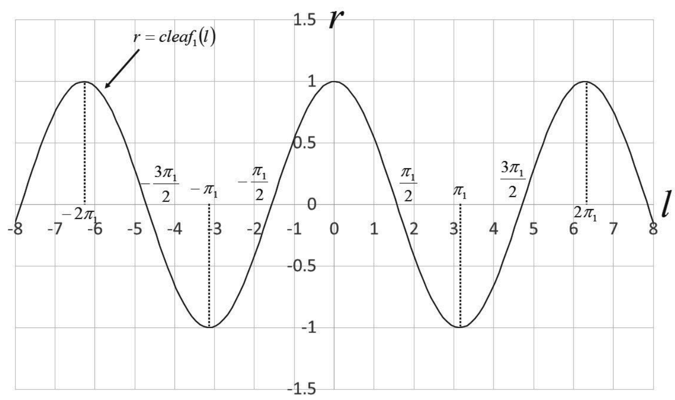

Fig. 14 Wave of leaf function $r=cleaf_1(l)$ $(=\cos(l))$ (1 period: $T=6.283(=2\pi_1)$)

Table 2 Relation between variables l and r for the leaf function: $r=cleaf_n(l)$ with respect to one period in both the positive ($0 \leqq l \leqq 2\pi_n$) and negative directions ($-2\pi_n \leqq l \leqq 0$)

Domain	Range of angle θ	Range of length l	Length l	Range of variable r	Derivation dr/dl
(1)	$-2\pi\frac{1}{n} \le \theta < -\frac{3}{2}\pi\frac{1}{n}$	$-2\pi_n \le l < -\frac{3}{2}\pi_n$	$l = -\frac{3\pi_n}{2} + \int_0^r \frac{-1}{\sqrt{1-t^{2n}}}dt$	$0 \leqq r \leqq 1$	$\frac{dr}{dl} = -\sqrt{1-r^{2n}}$
(2)	$-\frac{3}{2}\pi\frac{1}{n} \le \theta < -\pi\frac{1}{n}$	$-\frac{3}{2}\pi_n \le l < -\pi_n$	$l = -\pi_n + \int_{-1}^r \frac{-1}{\sqrt{1-t^{2n}}}dt$	$-1 \leqq r \leqq 0$	$\frac{dr}{dl} = -\sqrt{1-r^{2n}}$
(3)	$-\pi\frac{1}{n} \le \theta < -\frac{1}{2}\pi\frac{1}{n}$	$-\pi_n \le l < -\frac{1}{2}\pi_n$	$l = -\frac{\pi_n}{2} + \int_0^r \frac{1}{\sqrt{1-t^{2n}}}dt$	$-1 \leqq r \leqq 0$	$\frac{dr}{dl} = \sqrt{1-r^{2n}}$
(4)	$-\frac{1}{2}\pi\frac{1}{n} \le \theta < 0$	$-\frac{1}{2}\pi_n \le l < 0$	$l = \int_1^r \frac{1}{\sqrt{1-t^{2n}}}dt$	$0 \leqq r \leqq 1$	$\frac{dr}{dl} = \sqrt{1-r^{2n}}$
(5)	$0 \le \theta < \frac{1}{2}\pi\frac{1}{n}$	$0 \le l < \frac{1}{2}\pi_n$	$l = \int_1^r \frac{-1}{\sqrt{1-t^{2n}}}dt$	$0 \leqq r \leqq 1$	$\frac{dr}{dl} = -\sqrt{1-r^{2n}}$
(6)	$\frac{1}{2}\pi\frac{1}{n} \le \theta < \pi\frac{1}{n}$	$\frac{1}{2}\pi_n \le l < \pi_n$	$l = \frac{\pi_n}{2} + \int_0^r \frac{-1}{\sqrt{1-t^{2n}}}dt$	$-1 \leqq r \leqq 0$	$\frac{dr}{dl} = -\sqrt{1-r^{2n}}$
(7)	$\pi\frac{1}{n} \le \theta < \frac{3}{2}\pi\frac{1}{n}$	$\pi_n \le l < \frac{3}{2}\pi_n$	$l = \pi_n + \int_{-1}^r \frac{1}{\sqrt{1-t^{2n}}}dt$	$-1 \leqq r \leqq 0$	$\frac{dr}{dl} = \sqrt{1-r^{2n}}$
(8)	$\frac{3}{2}\pi\frac{1}{n} \le \theta < 2\pi\frac{1}{n}$	$\frac{3}{2}\pi_n \le l < 2\pi_n$	$l = \frac{3\pi_n}{2} + \int_0^r \frac{1}{\sqrt{1-t^{2n}}}dt$	$0 \leqq r \leqq 1$	$\frac{dr}{dl} = \sqrt{1-r^{2n}}$

Table 3 Relation between the variables r, l, and θ of the leaf function $cleaf_n(l)$

Range of angle θ	Range of length l	Length l	Range of variable r	Derivation dr/dl
$(2m-2)\pi\frac{1}{n} \le \theta < \left(2m-\frac{3}{2}\right)\pi\frac{1}{n}$	$(2m-2)\pi_n \le l < \left(2m-\frac{3}{2}\right)\pi_n$	$l = (2m-2)\pi_n + \int_1^r \frac{-1}{\sqrt{1-t^{2n}}}dt$	$0 \leqq r \leqq 1$	$\frac{dr}{dl} = -\sqrt{1-r^{2n}}$
$\left(2m-\frac{3}{2}\right)\pi\frac{1}{n} \le \theta < (2m-1)\pi\frac{1}{n}$	$\left(2m-\frac{3}{2}\right)\pi_n \le l < (2m-1)\pi_n$	$l = \left(2m-\frac{3}{2}\right)\pi_n + \int_0^r \frac{-1}{\sqrt{1-t^{2n}}}dt$	$-1 \leqq r \leqq 0$	$\frac{dr}{dl} = -\sqrt{1-r^{2n}}$
$(2m-1)\pi\frac{1}{n} \le \theta < \left(2m-\frac{1}{2}\right)\pi\frac{1}{n}$	$(2m-1)\pi_n \le l < \left(2m-\frac{1}{2}\right)\pi_n$	$l = (2m-1)\pi_n + \int_{-1}^r \frac{1}{\sqrt{1-t^{2n}}}dt$	$-1 \leqq r \leqq 0$	$\frac{dr}{dl} = \sqrt{1-r^{2n}}$
$\left(2m-\frac{1}{2}\right)\pi\frac{1}{n} \le \theta < 2m\pi\frac{1}{n}$	$\left(2m-\frac{1}{2}\right)\pi_n \le l < 2m\pi_n$	$l = \left(2m-\frac{1}{2}\right)\pi_n + \int_0^r \frac{1}{\sqrt{1-t^{2n}}}dt$	$0 \leqq r \leqq 1$	$\frac{dr}{dl} = \sqrt{1-r^{2n}}$

(Note) The number m represents the integer ($m=0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \cdots$)

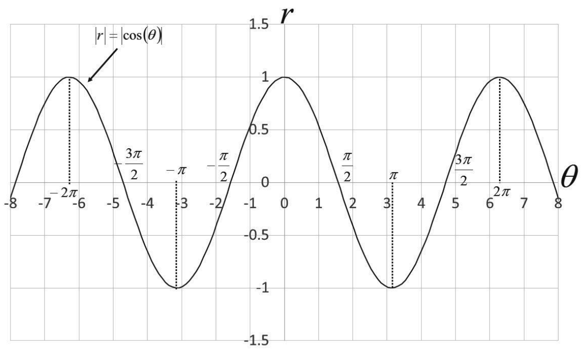

Fig. 15 Wave of leaf function $|r|=|cos(\theta)|$

(1 period: $T=\pi\times2$)

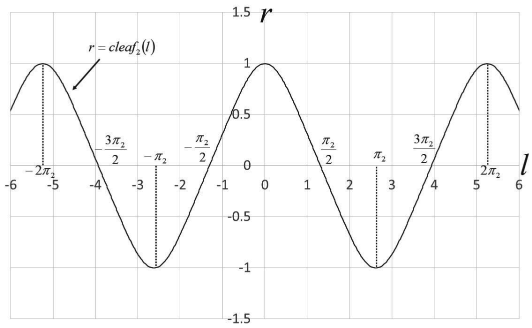

Fig. 16 Wave of leaf function $r=cleaf_2(l)$

(1 period: $T=5.244(=2\pi_2)$)

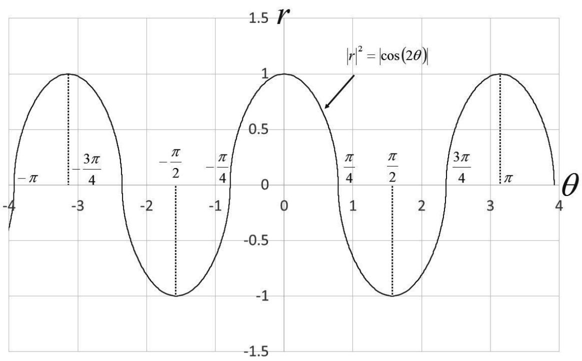

Fig. 17 Wave of leaf function $|r|^2=|cos(2\theta)|$

(1 period: $T=\pi/2\times2$)

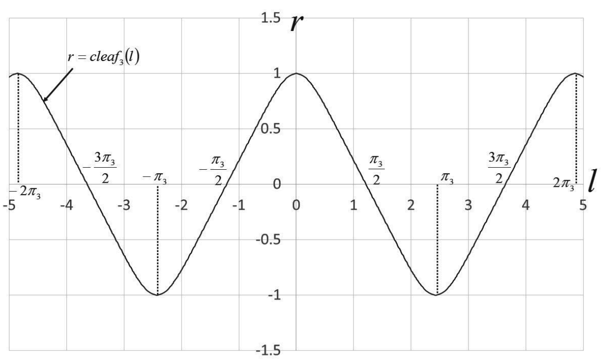

Fig. 18 Wave of leaf function $r=cleaf_3(l)$

(1 period: $T=4.857(=2\pi_3)$)

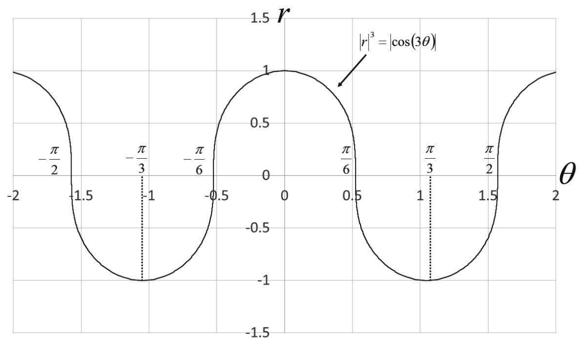

Fig. 19 Wave of leaf function $|r|^3=|cos(3\theta)|$

(1 period: $T=\pi/3\times2$)

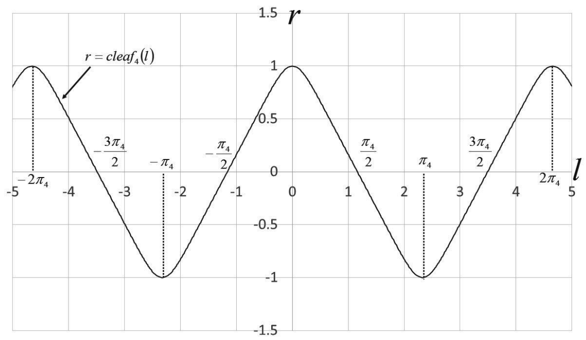

Fig. 20 Wave of leaf function $r=cleaf_4(l)$

(1 period: $T=4.654(=2\pi_4)$)

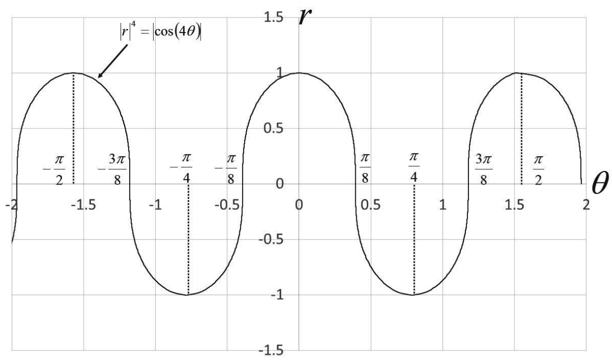

Fig. 21 Wave of leaf function $|r|^4=|cos(4\theta)|$

(1 period: $T=\pi/4\times2$)

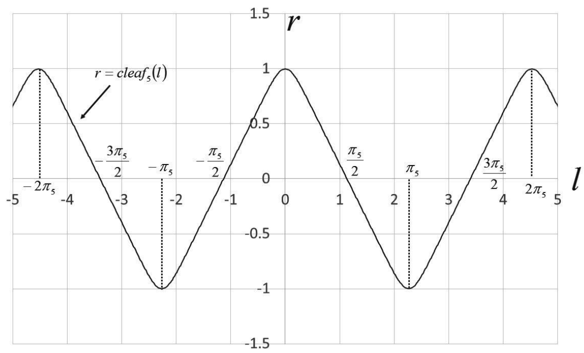

Fig. 22 Wave of leaf function $r=cleaf_5(l)$

(1 period: $T=4.529(=2\pi_5)$)

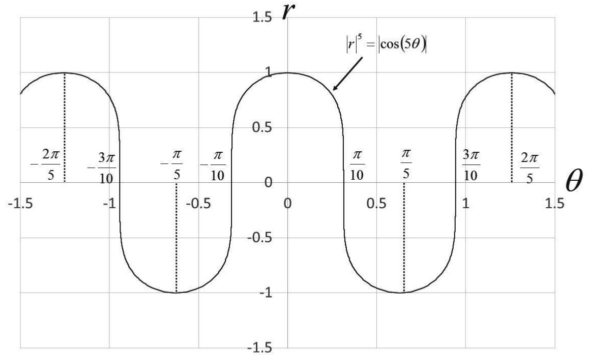

Fig. 23 Wave of leaf function $|r|^5=|cos(5\theta)|$
(1 period: $T=\pi/5\times2$)

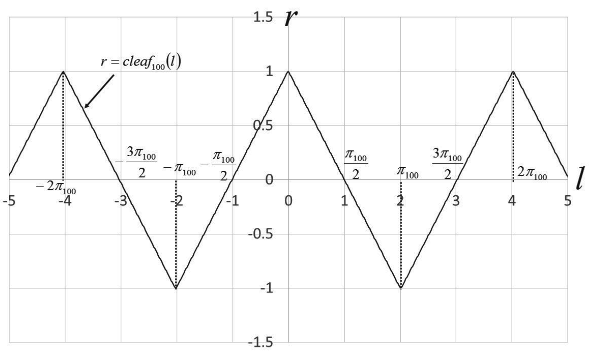

Fig. 24 Wave of leaf function $r=cleaf_{100}(l)$
(1 period: $T=4.028(=2\pi_{100})$)

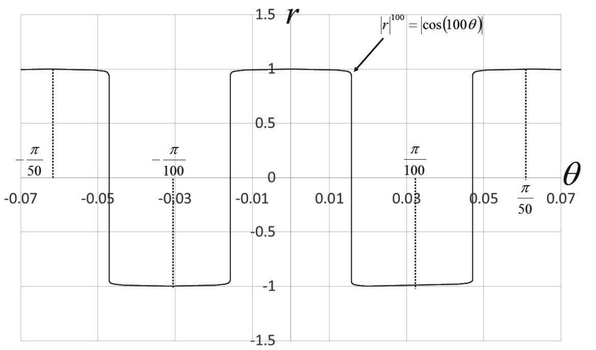

Fig. 25 Wave of leaf function $|r|^{100}=|cos(100\ \theta)|$
(1 period: $T=\pi/100\times2$)

5. Relation between the function *cleaf_n(l)* and the function *sleaf_n(l)*

Using Eq. (35) and Eq. (41), the leaf function: $cleaf_n(l)$ is obtained as follows:

$$|\cos n\theta| = |r|^n = |cleaf_n(l)|^n \tag{51}$$

The leaf function: $sleaf_n(l)$ is also obtained as follows:

$$|\sin n\theta| = |\bar{r}|^n = |sleaf_n(\bar{l})|^n \tag{52}$$

The variables $\bar{r}$ and $\bar{l}$ are described later. Using the trigonometric functions, the relation between $sin(n\theta)$ and $\cos(n\theta)$ is obtained as follows:

$$(\sin n\theta)^2 + (\cos n\theta)^2 = 1 \tag{53}$$

Using the above equation, with Eq. (51) and Eq. (52), the following equation is obtained:

$$\begin{aligned}(\sin n\theta)^2 + (\cos n\theta)^2 &= \left(\pm|sleaf_n(\bar{l})|^n\right)^2 + \left(\pm|cleaf_n(l)|^n\right)^2 \\ &= (cleaf_n(l))^{2n} + (sleaf_n(\bar{l}))^{2n} = 1\end{aligned} \tag{54}$$

As shown in Fig. 26, the variables l and $\bar{l}$ represent the length at the angle θ. With respect to the angle θ in the x-y graph, the arc length l of the function: $cleaf_n(l)$ is different from the arc length $\bar{l}$ of the function: $sleaf_n(\bar{l})$. The variable l in the leaf function $cleaf_n(l)$ takes the arc length between the coordinates $(x,y)=(1,0)$ and the point on the curve by the leaf function: $cleaf_n(l)$. On the other hand, the variable l in the leaf function $sleaf_n(l)$ takes the arc length between the coordinates $(x,y)=(0,0)$ and the point on the curve by the leaf function: $sleaf_n(l)$. For example, the case of $n=2$ (the lemniscate curve) is shown in Fig. 26. With respect to the angle $\theta=0$, the function $sleaf_n(\bar{l})$ takes $r=0$ at the point $(x,y)=(0,0)$. The function $cleaf_n(l)$ takes $r=1$ at the point $(x,y)=(1,0)$. When the angle θ increases, the arc length l increases. The ratio of increase in the leaf function $sleaf_n(\bar{l})$ is different from the ratio of increase in the leaf function $cleaf_n(l)$. Therefore, with respect to the arbitrary angle θ, The variables of the arc length l and $\bar{l}$ are not constantly satisfied with the equation $l=\bar{l}$.

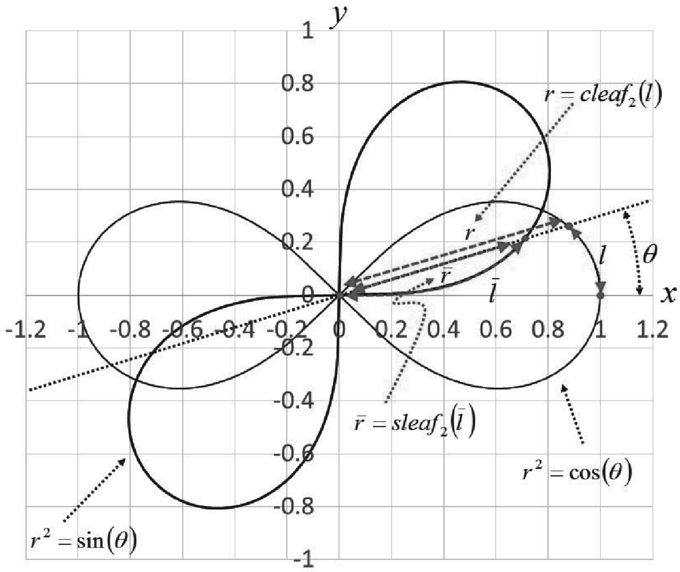

Fig. 26 Geometric relation between the leaf function: $cleaf_2(l)$ and the leaf function $sleaf_2(\bar{l})$

The variables in Eq. (54) consist of both the variables l and $\bar{l}$. We discuss the following equation.

$$\begin{aligned}&\frac{d}{dl}\arccos\left(cleaf_n(l)\right)^n\\&=-\frac{1}{\sqrt{1-\left(cleaf_n(l)\right)^{2n}}}n\left(cleaf_n(l)\right)^{n-1}\left\{-\sqrt{1-\left(cleaf_n(l)\right)^{2n}}\right\}\\&=n\left(cleaf_n(l)\right)^{n-1}\end{aligned}\tag{55}$$

The above equation is integrated from 0 to the variable l.

$$\left[\arccos\left(cleaf_n(t)\right)^n\right]_0^l=\int_0^l n\left(cleaf_n(t)\right)^{n-1}dt\tag{56}$$

$$\begin{aligned}\left[\arccos\left(cleaf_n(t)\right)^n\right]_0^l&=\arccos\left(cleaf_n(l)\right)^n-\arccos\left(cleaf_n(0)\right)^n\\&=\arccos\left(cleaf_n(l)\right)^n-\arccos(1)=\arccos\left(cleaf_n(l)\right)^n\end{aligned}\tag{57}$$

Therefore, it is obtained as follows:

$$\left(cleaf_n(l)\right)^n=\cos\left(n\int_0^l\left(cleaf_n(t)\right)^{n-1}dt\right)\tag{58}$$

Using the Eq. (8), the above equation and $r=cleaf_n(l)$, the angle θ in Fig.26 can be described as follows:

$$\theta=\int_0^l\left(cleaf_n(t)\right)^{n-1}dt\quad\left(=\frac{1}{n}\arccos\left(cleaf_n(l)\right)^n\right)\tag{59}$$

Next, we discuss the following equation with respect to the variable $\bar{l}$.

$$\begin{aligned}&\frac{d}{d\bar{l}}\arcsin\left(sleaf_n(\bar{l})\right)^n\\&=\frac{1}{\sqrt{1-\left(sleaf_n(\bar{l})\right)^{2n}}}n\left(sleaf_n(\bar{l})\right)^{n-1}\sqrt{1-\left(sleaf_n(\bar{l})\right)^{2n}}\\&=n\left(sleaf_n(\bar{l})\right)^{n-1}\end{aligned}\tag{60}$$

The above equation is integrated from 0 to the variable $\bar{l}$.

$$\left[\arcsin\left(sleaf_n(t)\right)^n\right]_0^{\bar{l}}=\int_0^{\bar{l}} n\left(sleaf_n(t)\right)^{n-1}dt\tag{61}$$

$$\begin{aligned}\left[\arcsin\left(sleaf_n(t)\right)^n\right]_0^{\bar{l}}&=\arcsin\left(sleaf_n(\bar{l})\right)^n-\arcsin\left(sleaf_n(0)\right)^n\\&=\arcsin\left(sleaf_n(\bar{l})\right)^n-\arcsin(0)=\arcsin\left(sleaf_n(\bar{l})\right)^n\end{aligned}\tag{62}$$

Therefore, it is obtained as follows:

$$\left(sleaf_n(\bar{l})\right)^n=\sin\left(n\int_0^{\bar{l}}\left(sleaf_n(t)\right)^{n-1}dt\right)\tag{63}$$

Using the Eq. (7), the above equation and $\bar{r}=sleaf_n(\bar{l})$, the angle θ in Fig.26 also can be described as follows:

$$\theta=\int_0^{\bar{l}}\left(sleaf_n(t)\right)^{n-1}dt\quad\left(=\frac{1}{n}\arcsin\left(sleaf_n(\bar{l})\right)^n\right)\tag{64}$$

However, Eq.(54) can be described by using only one variable l. In the case of $n=1$, the equation is obtained as follows:

$$\left(sleaf_1(l)\right)^2+\left(cleaf_1(l)\right)^2=1\tag{65}$$

The above equation is equal to the equation: $\left(\sin(l)\right)^2+\left(\cos(l)\right)^2=1$. In the case of $n=2$, the arbitrary variable l is satisfied with the following equation:

$$\left(sleaf_2(l)\right)^2+\left(cleaf_2(l)\right)^2+\left(sleaf_2(l)\right)^2\cdot\left(cleaf_2(l)\right)^2=1\tag{66}$$

In the case of $n=3$, the arbitrary variable l is satisfied with the following equation:

$$(sleaf_3(l))^2+(cleaf_3(l))^2+2\cdot(sleaf_3(l))^2\cdot(cleaf_3(l))^2=1 \tag{67}$$

(See proof in Appendix) Using the symmetry and the periodicity of waves in Figs. 14 - 25, the following equations are obtained:

$$sleaf_n(-l)=-sleaf_n(l) \tag{68}$$

$$cleaf_n(-l)=cleaf_n(l) \tag{69}$$

$$sleaf_n\left(\frac{\pi_n}{2}-l\right)=cleaf_n(l) \tag{70}$$

$$cleaf_n\left(\frac{\pi_n}{2}-l\right)=sleaf_n(l) \tag{71}$$

$$sleaf_n(\pi_n-l)=sleaf_n(l) \tag{72}$$

$$cleaf_n(\pi_n-l)=-cleaf_n(l) \tag{73}$$

$$sleaf_n\left(l+\frac{\pi_n}{2}\right)=cleaf_n(l) \tag{74}$$

$$cleaf_n\left(l+\frac{\pi_n}{2}\right)=-sleaf_n(l) \tag{75}$$

$$sleaf_n(l+\pi_n)=-sleaf_n(l) \tag{76}$$

$$cleaf_n(l+\pi_n)=-cleaf_n(l) \tag{77}$$

$$sleaf_n(l+2\pi_n)=sleaf_n(l) \tag{78}$$

$$cleaf_n(l+2\pi_n)=cleaf_n(l) \tag{79}$$

The following equations are obtained:

$$sleaf_n(m\pi_n)=0 \quad (m=0,\pm1,\pm2,\pm3,,,,,,,,,,,) \tag{80}$$

$$sleaf_n\left(\frac{\pi_n}{2}(4m-3)\right)=1 \quad (m=0,\ \pm1,\pm2,\pm3,,,,,,,,,,,) \tag{81}$$

$$sleaf_n\left(\frac{\pi_n}{2}(4m-1)\right)=-1 \quad (m=0,\ \pm1,\pm2,\pm3,,,,,,,,,,,) \tag{82}$$

$$cleaf_n\left(\frac{\pi_n}{2}(2m-1)\right)=0 \quad (m=0,\pm1,\pm2,\pm3,,,,,,,,,,,) \tag{83}$$

$$cleaf_n(2m\pi_n)=1 \quad (m=0,\pm1,\pm2,\pm3,,,,,,,,,,,) \tag{84}$$

$$cleaf_n(\pi_n(2m-1))=-1 \quad (m=0,\pm1,\pm2,\pm3,,,,,,,,,,,) \tag{85}$$

The constant π_n is obtained by Eq .(42).

6. Derivative of the leaf function

In this section, the derivative of the leaf function is described. As shown in Fig. 26, with respect to the angle θ, the length l and the length $\bar{l}$ represents the arc length l of the function: $cleaf_n(l)$ and the arc length $\bar{l}$ of the function: $sleaf_n(\bar{l})$. As shown in Fig. 26, the variable $\bar{l}$ depends on the length l. Therefore, we can regard the variable $\bar{l}$ as the function: $\bar{l}(l)$. The sign of the derivative of the leaf function depends on the range of the length l, and varies with respect to the range of the length l (See Table 2). We only discuss the range: $0\leqq l\leqq\pi_n/2$. Using the formula of the chain rule of differentiation, the following equation is obtained by differentiating Eq. (54) with respect to the variable l.

$$2n\cdot sleaf_n^{2n-1}(\bar{l})\cdot\sqrt{1-sleaf_n^{2n}(\bar{l})}\cdot\frac{d\bar{l}}{dl}+2n\cdot cleaf_n^{2n-1}(l)\cdot\left\{-\sqrt{1-cleaf_n^{2n}(l)}\right\}=0 \tag{86}$$

Using Eq. (54), the above equation is as follows:

$$2n\cdot sleaf_n^{2n-1}(\bar{l})\cdot cleaf_n^{n}(l)\cdot\frac{d\bar{l}}{dl}-2n\cdot cleaf_n^{2n-1}(l)\cdot sleaf_n^{n}(\bar{l})=0 \tag{87}$$

The above equation is as follows:

$$\frac{d\bar{l}}{dl}=\frac{cleaf_n^{n-1}(l)}{sleaf_n^{n-1}(\bar{l})} \tag{88}$$

The derivative of the leaf function: $cleaf_n(l)$ is obtained as follows:

$$\frac{d}{dl}cleaf_n(l)=-\sqrt{1-cleaf_n^{2n}(l)}=-sleaf_n^{n}(\bar{l}) \tag{89}$$

Eq. (54) is applied to the above equation. Note that the variable l is different from the variable $\bar{l}$. The second derivative of the leaf function: $cleaf_n(l)$ is obtained as follows:

$$\frac{d^2}{dl^2} cleaf_n(l) = -n \cdot sleaf_n^{n-1}(\bar{l}) \cdot \sqrt{1 - sleaf_n^{2n}(\bar{l})} \cdot \frac{d\bar{l}}{dl}$$
$$= -n \cdot cleaf_n^{2n-1}(l) \tag{90}$$
$$= -n \cdot sleaf_n^{n-1}(\bar{l}) \cdot cleaf_n^{n}(l) \cdot \frac{cleaf_n^{n-1}(l)}{sleaf_n^{n-1}(\bar{l})}$$
$$= -n \cdot cleaf_n^{2n-1}(l)$$

The third derivative of the leaf function: $cleaf_n(l)$ is obtained as follows:

$$\frac{d^3}{dl^3} cleaf_n(l)$$
$$= -n \cdot (2n-1) \cdot cleaf_n^{2n-2}(l) \cdot \left(-\sqrt{1 - cleaf_n^{2n}(l)}\right) \tag{91}$$
$$= n \cdot (2n-1) \cdot cleaf_n^{2n-2}(l) \cdot \sqrt{1 - cleaf_n^{2n}(l)}$$

The fourth derivative of the leaf function: $cleaf_n(l)$ is obtained as follows:

$$\frac{d^4}{dl^4} cleaf_n(l)$$
$$= -n \cdot (2n-1) \cdot (2n-2) \cdot cleaf_n^{2n-3}(l) \cdot \left(\sqrt{1 - cleaf_n^{2n}(l)}\right)^2$$
$$+ n \cdot (2n-1) \cdot cleaf_n^{2n-2}(l) \cdot \frac{-2n \cdot cleaf_n^{2n-1}(l)}{2\sqrt{1 - cleaf_n^{2n}(l)}} \sqrt{1 - cleaf_n^{2n}(l)} \tag{92}$$

The above equation is as follows:

$$\frac{d^4}{dl^4} cleaf_n(l) \tag{93}$$
$$= n \cdot (2n-1) \cdot cleaf_n^{2n-3}(l) \cdot \{2 - 2n + (-2 + 3n) \cdot cleaf_n^{2n}(l)\}$$

Next, we discuss the derivative of the leaf function: $sleaf_n(\bar{l})$. By differentiating Eq.(54) with respect to the variable $\bar{l}$, the following is obtained:

$$2n \cdot sleaf_n^{2n-1}(\bar{l}) \cdot \sqrt{1 - sleaf_n^{2n}(\bar{l})}$$
$$+ 2n \cdot cleaf_n^{2n-1}(l) \cdot \left\{-\sqrt{1 - cleaf_n^{2n}(l)}\right\} \cdot \frac{dl}{d\bar{l}} = 0 \tag{94}$$

The above equation is as follows:

$$\frac{dl}{d\bar{l}} = \frac{sleaf_n^{n-1}(\bar{l})}{cleaf_n^{n-1}(l)} \tag{95}$$

The first derivative of the leaf function: $sleaf_n(\bar{l})$ is obtained as follows:

$$\frac{d}{d\bar{l}} sleaf_n(\bar{l}) = \sqrt{1 - sleaf_n^{2n}(\bar{l})} = cleaf_n^{n}(l) \tag{96}$$

Note that the above equation is differentiated with respect to the variable $\bar{l}$. The second derivative of the leaf function $sleaf_n(\bar{l})$ is obtained as follows:

$$\frac{d^2}{d\bar{l}^2} sleaf_n(\bar{l})$$
$$= n \cdot cleaf_n^{n-1}(l) \cdot \left(-\sqrt{1 - cleaf_n^{2n}(l)}\right) \cdot \frac{dl}{d\bar{l}} \tag{97}$$
$$= n \cdot cleaf_n^{n-1}(l) \cdot \left(-sleaf_n^{n}(\bar{l})\right) \cdot \frac{sleaf_n^{n-1}(\bar{l})}{cleaf_n^{n-1}(l)}$$
$$= -n \cdot sleaf_n^{2n-1}(\bar{l})$$

The third derivative is obtained as follows:

$$\frac{d^3}{d\bar{l}^3} sleaf_n(\bar{l}) \tag{98}$$
$$= -n \cdot (2n-1) \cdot sleaf_n^{2n-2}(\bar{l}) \cdot \sqrt{1 - sleaf_n^{2n}(\bar{l})}$$

The fourth derivative is obtained as follows:

$$\frac{d^4}{d\bar{l}^4} sleaf_n(\bar{l})$$
$$= -n \cdot (2n-1) \cdot (2n-2) \cdot sleaf_n^{2n-3}(\bar{l}) \cdot \left(\sqrt{1 - sleaf_n^{2n}(\bar{l})}\right)^2$$
$$- n \cdot (2n-1) \cdot sleaf_n^{2n-2}(\bar{l}) \cdot \frac{-2n \cdot sleaf_n^{2n-1}(\bar{l})}{2\sqrt{1 - sleaf_n^{2n}(\bar{l})}} \sqrt{1 - sleaf_n^{2n}(\bar{l})} \tag{99}$$

The above equation is as follows:

$$\frac{d^4}{d\bar{l}^4} sleaf_n(\bar{l})$$
$$= n \cdot (2n-1) \cdot sleaf_n^{2n-3}(\bar{l}) \cdot \{2 - 2n + (3n-2) sleaf_n^{2n}(\bar{l})\} \tag{100}$$

7. Addition theorem of leaf function

The addition theorem of leaf functions is described. In the case of $n=1$ in Eq. (1), the functions that are satisfied with an ODE are the trigonometric functions: *sin(l)* and *cos(l)*. Using the leaf function, the addition theorem is described as follows:

$$sleaf_1(l_1 \pm l_2) = sleaf_1(l_1)cleaf_1(l_2) \pm sleaf_1(l_2)cleaf_1(l_1) \quad (101)$$
$$cleaf_1(l_1 \pm l_2) = cleaf_1(l_1)cleaf_1(l_2) \mp sleaf_1(l_1)sleaf_1(l_2) \quad (102)$$

In the case of $n=2$, the addition theorem of the leaf function is described based on the theorem of the elliptical function.

$$sleaf_2(l_1 \pm l_2) = \frac{sleaf_2(l_1)sleaf_2'(l_2) \pm sleaf_2'(l_1)sleaf_2(l_2)}{1+(sleaf_2(l_1))^2(sleaf_2(l_2))^2}$$
$$= \frac{sleaf_2(l_1)\sqrt{1-(sleaf_2(l_2))^4} \pm sleaf_2(l_2)\sqrt{1-(sleaf_2(l_1))^4}}{1+(sleaf_2(l_1))^2(sleaf_2(l_2))^2}$$

(103)

$$cleaf_2(l_1 \pm l_2) = \frac{cleaf_2(l_1)sleaf_2'(l_2) \pm cleaf_2'(l_1)sleaf_2(l_2)}{1+(cleaf_2(l_1))^2(sleaf_2(l_2))^2}$$
$$= \frac{cleaf_2(l_1)\sqrt{1-(sleaf_2(l_2))^4} \mp sleaf_2(l_2)\sqrt{1-(cleaf_2(l_1))^4}}{1+(cleaf_2(l_1))^2(sleaf_2(l_2))^2}$$

(104)

In the above equation, the superscript prime ’ of the leaf function represents the derivative with respect to the variable *l*. The sign of the derivative of the leaf function varied according to the range of the arc length *l*. As the range is $0 \leqq l \leqq \pi_n/2$, we discuss the above equation. In the other range of the variable *l*, given in Table 2 and Table 3, note the sign of the derivative of the leaf function.

8. Conclusion

In the first report, the leaf function $sleaf_n(l)$ is defined. In this report, the leaf function $cleaf_n(l)$ is defined. The relation between the leaf function: $cleaf_n(l)$ and the function: $sleaf_n(l)$ is presented.

References

[1]Paul F. Byrd and Morris D. Friedman: Handbook of Elliptic Integrals for Engineers and. Scientists, Second ed., Springer-Verlag, New York, 1971.
[2]Umberto Bottazzini and Jeremy Gray: Hidden Harmony - Geometric Fantasies. Springer, New York, 2013.
[3]J. Stillwell: Mathematics and Its History, Springer-Verlag, New York, 1989.
[4] A. C. Dixon: The Elementary Properties of the Elliptic Functions, with Examples, Macmillan, London, 1894.
[5] A. G. Greenhill: The Applications of Elliptic Functions, Macmillan, London, 1892.
[6] H. McKean and V. Moll: Elliptic Curves: Function Theory, Geometry and Arithmetic. Cambridge University, 1999.
[7] James Booth: The Theory of Elliptic Integrals, Book on Demand Ltd. , 2013.
[8] P. Franklin, W. E. Byerly and I. Todhunter: Elliptic Integrals - A Selection of Classic Mathematical Articles Containing Examples and Exercises on the Subject of Calculus (Mathematics Series), Burrard Press, 2012.
[9] N. I. Akhiezer: Elements of the Theory of Elliptic Functions (Translations of Mathematical Monographs), American Mathematical Society, 1990.
[10]Derek F. Lawden: Elliptic Functions and Applications (Applied Mathematical Sciences), Springer, 1989.

Appendix A

In the case of $n=2$ and 3 in Eq.(1), the Taylor expansion of the leaf functions are described in the appendix. These Taylor expansions are satisfied with Eq. (1), Eq. (66), and Eq. (67). The Taylor expansion of the leaf function is created by deriving the leaf function. First, in the case of $n=2$, the first derivative of the leaf function $sleaf_2(l)$ is obtained as follows:

$$\frac{d}{dl} sleaf_2(l) = \sqrt{1-sleaf_2^4(l)} \left(= cleaf_2^2(\bar{l}) = \frac{2cleaf_2(l)}{1+cleaf_2^2(l)} \right) \quad (A1)$$

The second derivative of the leaf function $sleaf_2(l)$ is obtained as follows:

$$\frac{d^2}{dl^2} sleaf_2(l) = -2 \cdot sleaf_2^3(l) \quad (A2)$$

The third derivative of the leaf function $sleaf_2(l)$ is obtained as follows:

$$\frac{d^3}{dl^3} sleaf_2(l) = -6 \cdot sleaf_2^2(l) \cdot \sqrt{1-sleaf_2^4(l)} \quad (A3)$$

The fourth derivative of the leaf function $sleaf_2(l)$ is obtained as follows:

$$\frac{d^4}{dl^4} sleaf_2(l) = 12 \cdot sleaf_2(l) \cdot \left(-1 + 2 sleaf_2^4(l)\right) \quad \text{(A4)}$$

The fifth derivative of the leaf function $sleaf_2(l)$ is obtained as follows:

$$\frac{d^5}{dl^5} sleaf_2(l) = 12 \cdot \left(-1 + 10 \cdot sleaf_2^4(l)\right)\sqrt{1 - sleaf_2^4(l)} \quad \text{(A5)}$$

The sixth derivative of the leaf function $sleaf_2(l)$ is obtained as follows:

$$\frac{d^6}{dl^6} sleaf_2(l) = 72 sleaf_2^3(l)\left(7 - 10 sleaf_2^4(l)\right) \quad \text{(A6)}$$

The seventh derivative of the leaf function $sleaf_2(l)$ is obtained as follows:

$$\frac{d^7}{dl^7} sleaf_2(l) = 504 sleaf_2^2(l)\left(3 - 10 sleaf_2^4(l)\right)\sqrt{1 - sleaf_2^4(l)} \quad \text{(A7)}$$

The eighth derivative of the leaf function $sleaf_2(l)$ is obtained as follows:

$$\frac{d^8}{dl^8} sleaf_2(l) = 1008 sleaf_2^2(l)\left(3 - 36 sleaf_2^4(l) + 40 sleaf_2^8(l)\right) \quad \text{(A8)}$$

The ninth derivative of the leaf function $sleaf_2(l)$ is obtained as follows:

$$\frac{d^9}{dl^9} sleaf_2(l) = 3024\left(1 - 60 sleaf_2^4(l) + 120 sleaf_2^8(l)\right)\sqrt{1 - sleaf_2^4(l)} \quad \text{(A9)}$$

The tenth derivative of the leaf function $sleaf_2(l)$ is obtained as follows:

$$\frac{d^{10}}{dl^{10}} sleaf_2(l) = -6048 sleaf_2^3(l)\left(121 - 660 sleaf_2^4(l) + 600 sleaf_2^8(l)\right) \quad \text{(A10)}$$

The eleventh derivative of the leaf function $sleaf_2(l)$ is obtained as follows:

$$\frac{d^{11}}{dl^{11}} sleaf_2(l) = -199584 sleaf_2^2(l)\left(11 - 140 sleaf_2^4(l) + 200 sleaf_2^8(l)\right)\sqrt{1 - sleaf_2^4(l)} \quad \text{(A11)}$$

The twelfth derivative of the leaf function $sleaf_2(l)$ is obtained as follows:

$$\frac{d^{12}}{dl^{12}} sleaf_2(l) = 399168\left(-11 sleaf_2(l) + 442 sleaf_2^5(l) - 1560 sleaf_2^9(l) + 1200 sleaf_2^{13}(l)\right) \quad \text{(A12)}$$

The thirteenth derivative of the leaf function $sleaf_2(l)$ is obtained as follows:

$$\frac{d^{13}}{dl^{13}} sleaf_2(l) = 399168\left\{-11 + 130 sleaf_2^4(l)\left(17 - 108 sleaf_2^4(l) + 120 sleaf_2^8(l)\right)\right\}\sqrt{1 - sleaf_2^4(l)} \quad \text{(A13)}$$

It continues in the same way below. The Taylor expansion is obtained as follows:

$$\begin{aligned} sleaf_2(l) &= sleaf_2(0) + \frac{1}{1!}\left(\frac{d}{dl} sleaf_2(0)\right)l \\ &+ \frac{1}{2!}\left(\frac{d^2}{dl^2} sleaf_2(0)\right)l^2 + \frac{1}{3!}\left(\frac{d^2}{dl^2} sleaf_2(0)\right)l^3 + \\ &\cdots\cdots + \frac{1}{13!}\left(\frac{d^{13}}{dl^{13}} sleaf_2(0)\right)l^{13} + O\left(l^{17}\right) \\ &= \frac{1}{1!}l - \frac{12}{5!}l^5 + \frac{3024}{9!}l^9 - \frac{4390848}{13!}l^{13} + O\left(l^{17}\right) \\ &= l - \frac{1}{10}l^5 + \frac{1}{120}l^9 - \frac{11}{15600}l^{13} + O\left(l^{17}\right) \end{aligned} \quad \text{(A14)}$$

The symbol O represents the Landau symbol. The symbol $O(l^{17})$ represents the order of the error.

The difference: $sleaf_2(l) - \left(l - \frac{1}{10}l^5 + \frac{1}{120}l^9 - \frac{11}{15600}l^{13}\right)$ is within $|l|^{17}$ when the variable l is sufficiently close to 0. The polynomial of Eq. (A14) is differentiated as follows:

$$\frac{d^2}{dl^2} sleaf_2(l) = -2l^3 + \frac{3}{5}l^7 - \frac{11}{100}l^{11} + O\left(l^{15}\right) \quad \text{(A15)}$$

Using Eq. (A14), the equation is obtained as follows:

$$
\begin{aligned}
&-2\cdot\left(sleaf_2(l)\right)^3\\
&=-2\cdot\left(l-\frac{1}{10}l^5+\frac{1}{120}l^9-\frac{11}{15600}l^{13}+O\left(l^{17}\right)\right)^3\\
&=-2l^3+\frac{3}{5}l^7-\frac{11}{100}l^{11}+O\left(l^{15}\right)
\end{aligned} \tag{A16}
$$

Through the results of both Eq. (A15) and Eq. (A16), the leaf function: *sleaf₂(l)* is satisfied with Eq. (1). Next, the Taylor expansion is applied to the leaf function: *cleaf₂(l)*. The first derivative of the leaf function *cleaf₂(l)* is obtained as follows:

$$
\begin{aligned}
&\frac{d}{dl}cleaf_2(l)\\
&=-\sqrt{1-cleaf_2^4(l)}\left(=-sleaf_2^2(\bar{l})=-\frac{2sleaf_2(l)}{1+sleaf_2^2(l)}\right)
\end{aligned} \tag{A17}
$$

The second derivative of the leaf function *cleaf₂(l)* is obtained as follows:

$$
\frac{d^2}{dl^2}cleaf_2(l)=-2\cdot cleaf_2^3(l) \tag{A18}
$$

The third derivative of the leaf function *cleaf₂(l)* is obtained as follows:

$$
\frac{d^3}{dl^3}cleaf_2(l)=6\cdot cleaf_2^2(l)\cdot\sqrt{1-cleaf_2^4(l)} \tag{A19}
$$

The fourth derivative of the leaf function *cleaf₂(l)* is obtained as follows:

$$
\frac{d^4}{dl^4}cleaf_2(l)=12\cdot cleaf_2(l)\cdot\left(-1+2cleaf_2^4(l)\right) \tag{A20}
$$

The fifth derivative of the leaf function *cleaf₂(l)* is obtained as follows:

$$
\frac{d^5}{dl^5}cleaf_2(l)=12\left(1-10cleaf_2^4(l)\right)\sqrt{1-cleaf_2^4(l)} \tag{A21}
$$

The sixth derivative of the leaf function *cleaf₂(l)* is obtained as follows:

$$
\frac{d^6}{dl^6}cleaf_2(l)=72cleaf_2^3(l)\left(7-10cleaf_2^4(l)\right) \tag{A22}
$$

The seventh derivative of the leaf function *cleaf₂(l)* is obtained as follows:

$$
\begin{aligned}
&\frac{d^7}{dl^7}cleaf_2(l)\\
&=504cleaf_2^2(l)\left(-3+10cleaf_2^4(l)\right)\sqrt{1-cleaf_2^4(l)}
\end{aligned} \tag{A23}
$$

The eighth derivative of the leaf function *cleaf₂(l)* is obtained as follows:

$$
\frac{d^8}{dl^8}cleaf_2(l)=1008cleaf_2(l)\left(3-36cleaf_2^4(l)+40cleaf_2^8(l)\right) \tag{A24}
$$

The ninth derivative of the leaf function *cleaf₂(l)* is obtained as follows:

$$
\begin{aligned}
&\frac{d^9}{dl^9}cleaf_2(l)\\
&=-3024\sqrt{1-cleaf_2^4(l)}\left(1-60cleaf_2^4(l)+120cleaf_2^8(l)\right)
\end{aligned} \tag{A25}
$$

The tenth derivative of the leaf function *cleaf₂(l)* is obtained as follows:

$$
\begin{aligned}
&\frac{d^{10}}{dl^{10}}cleaf_2(l)\\
&=-6048cleaf_2^3(l)\left(121-660cleaf_2^4(l)+600cleaf_2^8(l)\right)
\end{aligned} \tag{A26}
$$

The eleventh derivative of the leaf function *cleaf₂(l)* is obtained as follows:

$$
\begin{aligned}
&\frac{d^{11}}{dl^{11}}cleaf_2(\bar{l})\\
&=199584cleaf_2^2(l)\left(11-140cleaf_2^4(l)+200cleaf_2^8(l)\right)\sqrt{1-cleaf_2^4(l)}
\end{aligned} \tag{A27}
$$

The twelfth derivative of the leaf function *cleaf₂(l)* is obtained as follows:

$$
\begin{aligned}
&\frac{d^{12}}{dl^{12}}cleaf_2(l)\\
&=399168cleaf_2(l)\left\{-11+2cleaf_2^4(l)\left(221-780cleaf_2^4(l)+600cleaf_2^8(l)\right)\right\}
\end{aligned} \tag{A28}
$$

It continues in the same way below. The Taylor expansion is obtained as follows:

$$\begin{aligned}
cleaf_2(l) &= cleaf_2(0)+\frac{1}{1!}\left(\frac{d}{dl}cleaf_2(0)\right)l \\
&+\frac{1}{2!}\left(\frac{d^2}{dl^2}cleaf_2(0)\right)l^2+\frac{1}{3!}\left(\frac{d^2}{dl^2}cleaf_2(0)\right)l^3+ \\
&\cdots\cdots+\frac{1}{12!}\left(\frac{d^{12}}{dl^{12}}cleaf_2(0)\right)l^{12}+O\left(l^{13}\right) \\
&=1-\frac{2}{2!}l^2+\frac{12}{4!}l^4-\frac{216}{6!}l^6+\frac{7056}{8!}l^8 \\
&-\frac{368928}{10!}l^{10}+\frac{28340928}{12!}l^{12}+O\left(l^{14}\right) \\
&=1-l^2+\frac{1}{2}l^4-\frac{3}{10}l^6+\frac{7}{40}l^8 \\
&-\frac{61}{600}l^{10}+\frac{71}{1200}l^{12}+O\left(l^{14}\right)
\end{aligned} \tag{A29}$$

Using the above polynomial, the following equation is obtained:

$$\begin{aligned}
\frac{d^2}{dl^2}cleaf_2(l) &= -2+6l^2-9l^4+\frac{49}{5}l^6 \\
&-\frac{183}{20}l^8+\frac{781}{100}l^{10}-\frac{1253}{200}l^{12}+O\left(l^{14}\right)
\end{aligned} \tag{A30}$$

The following equation is obtained by Eq. (A29).

$$\begin{aligned}
&-2\cdot\left(cleaf_2(l)\right)^3 \\
&=-2\cdot\left(1-l^2+\frac{1}{2}l^4-\frac{3}{10}l^6+\frac{7}{40}l^8-\frac{61}{600}l^{10}+\frac{71}{1200}l^{12}+O\left(l^{14}\right)\right)^3 \\
&=-2+6l^2-9l^4+\frac{49}{5}l^6-\frac{183}{20}l^8+\frac{781}{100}l^{10}-\frac{1253}{200}l^{12}+O\left(l^{14}\right)
\end{aligned} \tag{A31}$$

By both Eq. (A30) and Eq. (A31), we find that the Taylor expansion of the leaf function: *cleaf₂(l)* is satisfied with Eq.(1). On the other hand, by substituting Eqs. (A32)-(A34) to Eq. (66), all terms are cancelled except for “ 1 ”.

$$\begin{aligned}
&\left(cleaf_2(l)\right)^2 \\
&=\left(1-l^2+\frac{1}{2}l^4-\frac{3}{10}l^6+\frac{7}{40}l^8-\frac{61}{600}l^{10}+\frac{71}{1200}l^{12}+O\left(l^{14}\right)\right)^2 \\
&=1-2l^2+2l^4-\frac{8}{5}l^6+\frac{6}{5}l^8-\frac{64}{75}l^{10}+\frac{44}{75}l^{12}+O\left(l^{14}\right)
\end{aligned} \tag{A32}$$

$$\begin{aligned}
\left(sleaf_2(l)\right)^2 &=\left(l-\frac{1}{10}l^5+\frac{1}{120}l^9-\frac{11}{15600}l^{13}+O\left(l^{15}\right)\right)^2 \\
&=l^2-\frac{1}{5}l^6+\frac{2}{75}l^{10}-\frac{1}{325}l^{14}+O\left(l^{18}\right)
\end{aligned} \tag{A33}$$

$$\begin{aligned}
&\left(sleaf_2(l)\right)^2\cdot\left(cleaf_2(l)\right)^2 \\
&=l^2-2l^4+\frac{9}{5}l^6-\frac{6}{5}l^8+\frac{62}{75}l^{10}-\frac{44}{75}l^{12}+O\left(l^{14}\right)
\end{aligned} \tag{A34}$$

Therefore, a Taylor expansion can be used to satisfy these equations with Eq. (66).

Next, in the case of *n=3*, the Taylor expansion is applied to the leaf function. The first derivative of the leaf function *sleaf₃(l)* is obtained as follows:

$$\frac{d}{dl}sleaf_3(l)=\sqrt{1-sleaf_3^6(l)} \tag{A35}$$

The second derivative of the leaf function *sleaf₃(l)* is obtained as follows:

$$\frac{d^2}{dl^2}sleaf_3(l)=-3\cdot sleaf_3^5(l) \tag{A36}$$

The third derivative of the leaf function *sleaf₃(l)* is obtained as follows:

$$\frac{d^3}{dl^3}sleaf_3(l)=-15\cdot sleaf_3^4(l)\cdot\sqrt{1-sleaf_3^6(l)} \tag{A37}$$

The fourth derivative of the leaf function *sleaf₃(l)* is obtained as follows:

$$\frac{d^4}{dl^4}sleaf_3(l)=15\cdot sleaf_3^3(l)\cdot\left(-4+7sleaf_3^6(l)\right) \tag{A38}$$

The fifth derivative of the leaf function *sleaf₃(l)* is obtained as follows:

$$\frac{d^5}{dl^5}sleaf_3(l)=45sleaf_3^2(l)\left(-4+21sleaf_3^6(l)\right)\sqrt{1-sleaf_3^6(l)} \tag{A39}$$

The sixth derivative of the leaf function *sleaf₃(l)* is obtained as follows:

$$\frac{d^6}{dl^6} sleaf_3(l) = -45 sleaf_3(l)\left(8 - 188 sleaf_3^6(l) + 231 sleaf_3^{12}(l)\right) \tag{A40}$$

The seventh derivative of the leaf function *sleaf3(l)* is obtained as follows:

$$\frac{d^7}{dl^7} sleaf_3(l) = -45\left\{8 + 7 sleaf_3^6(l)\left(-188 + 429 sleaf_3^6(l)\right)\right\}\sqrt{1 - sleaf_3^6(l)} \tag{A41}$$

The eighth derivative of the leaf function *sleaf3(l)* is obtained as follows:

$$\frac{d^8}{dl^8} sleaf_3(l) = 2025 sleaf_3^5(l)\left\{176 + 7 sleaf_3^6(l)\left(-152 + 143 sleaf_3^6(l)\right)\right\} \tag{A42}$$

The ninth derivative of the leaf function *sleaf3(l)* is obtained as follows:

$$\frac{d^9}{dl^9} sleaf_3(l) = 22275 sleaf_3^4(l)\left\{80 + 7 sleaf_3^6(l)\left(-152 + 221 sleaf_3^6(l)\right)\right\}\sqrt{1 - sleaf_3^6(l)} \tag{A43}$$

The tenth derivative of the leaf function *sleaf3(l)* is obtained as follows:

$$\frac{d^{10}}{dl^{10}} sleaf_3(l) = -22275 sleaf_3^3(l)\left\{-320 + 7 sleaf_3^6(l)\left(1600 - 5512 sleaf_3^6(l) + 4199 sleaf_3^{12}(l)\right)\right\} \tag{A44}$$

The eleventh derivative of the leaf function *sleaf3(l)* is obtained as follows:

$$\frac{d^{11}}{dl^{11}} sleaf_3(l) = -66825 sleaf_3^2(l)\sqrt{1 - sleaf_3^6(l)} \times\left\{-320 + 7 sleaf_2^6(l)\left(4800 - 27560 sleaf_2^6(l) + 29393 sleaf_2^{12}(l)\right)\right\} \tag{A45}$$

The twelfth derivative of the leaf function *sleaf3(l)* is obtained as follows:

$$\frac{d^{12}}{dl^{12}} sleaf_3(l) = 42768000\ sleaf_3(l) - 1806948000\ 0\ sleaf_3^7(l) + 2051848260\ 00\ sleaf_3^{13}(l) - 4941481545\ 00\ sleaf_3^{19}(l) + 3162341432\ 25\ sleaf_3^{25}(l) \tag{A46}$$

The thirteenth derivative of the leaf function *sleaf3(l)* is obtained as follows:

$$\frac{d^{13}}{dl^{13}} sleaf_3(l) = 334125\sqrt{1 - sleaf_3^6(l)} \times (128 - 378560\ sleaf_3^6(l) + 7983248\ sleaf_3^{12}(l) - 28099708\ sleaf_3^{18}(l) + 23661365\ sleaf_3^{24}(l)) \tag{A47}$$

It continues in the same way below. Using the Taylor expansion, the polynomial is obtained as follows:

$$\begin{aligned} sleaf_3(l) &= \frac{1}{1!}l - \frac{360}{7!}l^7 + \frac{42768000}{13!}l^{13} - \frac{9108557568\ 0000}{19!}l^{19} + O(l^{25}) \\ &= l - \frac{1}{14}l^7 + \frac{5}{728}l^{13} - \frac{145}{193648}l^{19} + \frac{4663}{54221440}l^{25} - \frac{3111273}{3059173644\ 80}l^{31} \\ &+ \frac{77686677}{6338607792\ 0256}l^{37} + O(l^{43}) \end{aligned} \tag{A48}$$

Using the above polynomial, the following equation is obtained as follows:

$$\frac{d^2}{dl^2} sleaf_3(l) = -3l^5 + \frac{15}{14}l^7 - \frac{1305}{5096}l^{17} + O(l^{23}) \tag{A49}$$

Using Eq.(A48), the following equation is obtained:

$$\begin{aligned} &-3\cdot\left(sleaf_3(l)\right)^5 \\ &= -3\cdot\left(l - \frac{1}{14}l^7 + \frac{5}{728}l^{13} - \frac{145}{193648}l^{19} + O(l^{25})\right)^5 \\ &= -3l^5 + \frac{15}{14}l^7 - \frac{1305}{5096}l^{17} + O(l^{23}) \end{aligned} \tag{A50}$$

Next, the Taylor expansion is applied to the leaf function: *cleaf3(l)*. The first derivative of the leaf function *cleaf3(l)* is obtained as follows:

$$\frac{d}{dl} cleaf_3(l) = -\sqrt{1 - cleaf_3^6(l)} \tag{A51}$$

The second derivative of the leaf function *cleaf₃(l)* is obtained as follows:

$$\frac{d^2}{dl^2} cleaf_3(l) = -3 \cdot cleaf_3^5(l) \tag{A52}$$

The third derivative of the leaf function $cleaf_3(l)$ is obtained as follows:

$$\frac{d^3}{dl^3} cleaf_3(l) = 15 \cdot cleaf_3^4(l) \cdot \sqrt{1 - cleaf_3^6(l)} \tag{A53}$$

The fourth derivative of the leaf function $cleaf_3(l)$ is obtained as follows:

$$\frac{d^4}{dl^4} cleaf_3(l) = 15 \cdot cleaf_3^3(l) \cdot \left(-4 + 7 cleaf_3^6(l)\right) \tag{A54}$$

The fifth derivative of the leaf function $cleaf_3(l)$ is obtained as follows:

$$\frac{d^5}{dl^5} cleaf_3(l) = 45 cleaf_3^2(l)\left(4 - 21 cleaf_3^6(l)\right)\sqrt{1 - cleaf_3^6(l)} \tag{A55}$$

The sixth derivative of the leaf function $cleaf_3(l)$ is obtained as follows:

$$\frac{d^6}{dl^6} cleaf_3(l) = -45 cleaf_3\ (l)\left(8 - 188 cleaf_3^6(l) + 231 cleaf_3^{12}(l)\right) \tag{A56}$$

The seventh derivative of the leaf function $cleaf_3(l)$ is obtained as follows:

$$\frac{d^7}{dl^7} cleaf_3(l) = 45 sleaf_3^3(\bar{l})\left\{8 + 7 cleaf_3^6(l)\left(-188 + 429 cleaf_3^6(l)\right)\right\} \tag{A57}$$

The eighth derivative of the leaf function $cleaf_3(l)$ is obtained as follows:

$$\frac{d^8}{dl^8} cleaf_3(l) = 2025 cleaf_3^5(l)\left\{176 + 7 cleaf_3^6(l)\left(-152 + 143 cleaf_3^6(l)\right)\right\} \tag{A58}$$

The ninth derivative of the leaf function $cleaf_3(l)$ is obtained as follows:

$$\frac{d^9}{dl^9} cleaf_3(l) = -22275 cleaf_3^4(l)\left\{80 + 7 cleaf_3^6(l)\left(-152 + 221 cleaf_3^6(l)\right)\right\}\sqrt{1 - cleaf_3^6(l)} \tag{A59}$$

The tenth derivative of the leaf function $cleaf_3(l)$ is obtained as follows:

$$\frac{d^{10}}{dl^{10}} cleaf_3(l) = -22275 cleaf_3^3(l) \cdot \left\{-320 + 7 cleaf_3^6(l)\left(1600 - 5512 cleaf_3^6(l) + 4199 cleaf_3^{12}(l)\right)\right\} \tag{A60}$$

The eleventh derivative of the leaf function $cleaf_3(l)$ is obtained as follows:

$$\frac{d^{11}}{dl^{11}} cleaf_3(l) = 66825 cleaf_3^2(l)\sqrt{1 - cleaf_3^6(l)} \cdot \left\{-320 + 7 cleaf_3^6(l)\left(4800 - 27560 cleaf_3^6(l) + 29393 cleaf_3^{12}(l)\right)\right\} \tag{A61}$$

The twelfth derivative of the leaf function $cleaf_3(l)$ is obtained as follows:

$$\frac{d^{12}}{dl^{12}} cleaf_3(l) = 42768000\ cleaf_3\ (l) - 1806948000\ 0 cleaf_3^7(l) + 2051848260\ 00\ cleaf_3^{13}(l) - 4941481545\ 00\ cleaf_3^{19}(l) + 3162341432\ 25\ cleaf_3^{25}(l) \tag{A62}$$

It continues in the same way below. Using the Taylor expansion, the polynomial is obtained as follows:

$$
\begin{aligned}
cleaf_3(l) &= 1-\frac{3}{2!}l^2+\frac{45}{4!}l^4-\frac{2295}{6!}l^6 \\
&+\frac{228825}{8!}l^8-\frac{37622475}{10!}l^{10}+\frac{9244102725}{12!}l^{12} \\
&-\frac{3173993373\ 375}{14!}l^{14}+O\left(l^{16}\right) \\
&=1-\frac{3}{2}l^2+\frac{15}{8}l^4-\frac{51}{16}l^6 \\
&+\frac{5085}{896}l^8-\frac{18579}{1792}l^{10}+\frac{138333}{7168}l^{12} \\
&-\frac{47497095}{1304576}l^{14}+O\left(l^{16}\right)
\end{aligned} \tag{A63}
$$

Using Eq. (A63), the following equation is obtained:

$$
\begin{aligned}
\frac{d^2}{dl^2}cleaf_3(l) &= -3+\frac{45}{2}l^2-\frac{765}{8}l^4+\frac{5085}{16}l^6 \\
&-\frac{836055}{896}l^8+\frac{4564989}{1792}l^{10}+O\left(l^{12}\right)
\end{aligned} \tag{A64}
$$

Using Eq. (A63), the following equation is obtained:

$$
\begin{aligned}
&-3\cdot\left(cleaf_3(l)\right)^5 \\
&=-3\cdot\left(1-\frac{3}{2}l^2+\frac{15}{8}l^4-\frac{51}{16}l^6+\frac{5085}{896}l^8-\frac{18579}{1792}l^{10}+O\left(l^{12}\right)\right)^5 \\
&=-3+\frac{45}{2}l^2-\frac{765}{8}l^4+\frac{5085}{16}l^6 \\
&-\frac{836055}{896}l^8+\frac{4564989}{1792}l^{10}+O\left(l^{12}\right)
\end{aligned} \tag{A65}
$$

By Eq.(A64) and Eq. (A65), the polynomial of the leaf function by Taylor is satisfied with Eq. (1). The following equation is obtained by substituting the polynomial in Eq. (67).

$$
\begin{aligned}
&\left(cleaf_3(l)\right)^2 \\
&=\left(1-\frac{3}{2}l^2+\frac{15}{8}l^4-\frac{51}{16}l^6+\frac{5085}{896}l^8-\frac{18579}{1792}l^{10}+\frac{138333}{7168}l^{12}+O\left(l^{14}\right)\right)^2 \\
&=1-3l^2+6l^4-12l^6+\frac{171}{7}l^8-\frac{348}{7}l^{10}+\frac{708}{7}l^{12}+O\left(l^{14}\right)
\end{aligned} \tag{A66}
$$

$$
\begin{aligned}
\left(sleaf_3(l)\right)^2 &= \left(l-\frac{1}{14}l^7+\frac{5}{728}l^{13}-\frac{145}{193648}l^{19}+O\left(l^{25}\right)\right)^2 \\
&=l^2-\frac{1}{7}l^8+\frac{12}{637}l^{14}-\frac{30}{12103}l^{20}+\frac{10865}{70487872}l^{26}+O\left(l^{32}\right)
\end{aligned} \tag{A67}
$$

$$
\begin{aligned}
&2\left(sleaf_3(l)\right)^2\cdot\left(cleaf_3(l)\right)^2 \\
&=2l^2-6l^4+12l^6-\frac{170}{7}l^8+\frac{348}{7}l^{10}-\frac{708}{7}l^{12}+O\left(l^{14}\right)
\end{aligned} \tag{A68}
$$

By substituting Eqs. (A66)-(A68) in Eq. (67), all terms are cancelled except for “ 1 ”. Therefore, a Taylor expansion can be used to satisfy these equations with Eq. (67).

Next, in the case of $n=4$, the Taylor expansion is applied to the leaf function. The first derivative of the leaf function $sleaf_4(l)$ is obtained as follows:

$$
\frac{d}{dl}sleaf_4(l)=\sqrt{1-sleaf_4^{\,8}(l)}=cleaf_4^{\,4}(\bar{l}) \tag{A69}
$$

The second derivative of the leaf function $sleaf_4(l)$ is obtained as follows:

$$
\frac{d^2}{dl^2}sleaf_4(l)=-4\cdot sleaf_4^{\,7}(l) \tag{A70}
$$

The third derivative of the leaf function $sleaf_4(l)$ is obtained as follows:

$$
\frac{d^3}{dl^3}sleaf_4(l)=-28\cdot sleaf_4^{\,6}(l)\cdot\sqrt{1-sleaf_4^{\,8}(l)} \tag{A71}
$$

The fourth derivative of the leaf function $sleaf_4(l)$ is obtained as follows:

$$
\frac{d^4}{dl^4}sleaf_4(l)=56\cdot sleaf_4^{\,5}(l)\cdot\left(-3+5sleaf_4^{\,8}(l)\right) \tag{A72}
$$

The fifth derivative of the leaf function $sleaf_4(l)$ is obtained as follows:

$$
\frac{d^5}{dl^5}sleaf_4(l)=280\,sleaf_4^{\,4}(l)\left(-3+13sleaf_4^{\,8}(l)\right)\sqrt{1-sleaf_4^{\,8}(l)} \tag{A73}
$$

The sixth derivative of the leaf function $sleaf_4(l)$ is obtained as follows:

$$
\frac{d^6}{dl^6}sleaf_4(l)=-1120\,sleaf_4^{\,3}(l)\left(3-45sleaf_4^{\,8}(l)+52sleaf_4^{\,16}(l)\right) \tag{A74}
$$

The seventh derivative of the leaf function $sleaf_4(l)$ is obtained as follows:

$$\frac{d^7}{dl^7} sleaf_4(l) = -1120\, sleaf_4^2(l)\sqrt{1 - sleaf_4^8(l)} \cdot \left(9 - 495\, sleaf_4^8(l) + 988\, sleaf_4^{16}(l)\right) \tag{A75}$$

The eighth derivative of the leaf function *sleaf₄(l)* is obtained as follows:

$$\frac{d^8}{dl^8} sleaf_4(l) = 2240\, sleaf_4(l) \cdot \left(-9 + 2502\, sleaf_4^8(l) - 12357\, sleaf_4^{16}(l) + 10868\, sleaf_4^{24}(l)\right) \tag{A76}$$

The ninth derivative of the leaf function $sleaf_4(l)$ is obtained as follows:

$$\frac{d^9}{dl^9} sleaf_4(l) = 2240\sqrt{1 - sleaf_4^8(l)} \cdot \left(-9 + 22518\, sleaf_4^8(l) - 210069\, sleaf_4^{16}(l) + 271700\, sleaf_4^{24}(l)\right) \tag{A77}$$

The tenth derivative of the leaf function $sleaf_4(l)$ is obtained as follows:

$$\frac{d^{10}}{dl^{10}} sleaf_4(l) = -313600\, sleaf_4^7(l) \cdot \left(-1287 + 25938\, sleaf_4^8(l) - 76587\, sleaf_4^{16}(l) + 54340\, sleaf_4^{24}(l)\right) \tag{A78}$$

The eleventh derivative of the leaf function $sleaf_4(l)$ is obtained as follows:

$$\frac{d^{11}}{dl^{11}} sleaf_4(l) = -313600\, sleaf_4^6(l)\sqrt{1 - sleaf_4^8(l)} \cdot \left(-9009 + 389070\, sleaf_4^8(l) - 1761501\, sleaf_4^{16}(l) + 1684540\, sleaf_4^{24}(l)\right) \tag{A79}$$

The twelfth derivative of the leaf function $sleaf_4(l)$ is obtained as follows:

$$\frac{d^{12}}{dl^{12}} sleaf_4(l) = 627200\, sleaf_4^5(l) \cdot \left[27027 + 17\, sleaf_4^8(l)\left\{-162855 + 13\, sleaf_4^8(l)\left(103521 - 217953\, sleaf_4^8(l) + 129580\, sleaf_4^{16}(l)\right)\right\}\right] \tag{A80}$$

The thirteenth derivative of the leaf function $sleaf_4(l)$ is obtained as follows:

$$\frac{d^{13}}{dl^{13}} sleaf_4(l) = 8153600\, sleaf_4^4(l)\sqrt{1 - sleaf_4^8(l)} \cdot \left\{10395 + 17\, sleaf_4^8(l)\left(-162855 + 2173941\, sleaf_4^8(l) - 6320637\, sleaf_4^{16}(l) + 4794460\, sleaf_4^{24}(l)\right)\right\} \tag{A81}$$

It continues in the same way below. Using the Taylor expansion, the polynomial is obtained as follows:

$$sleaf_4(l) = l - \frac{1}{18} l^9 + \frac{7}{1224} l^{17} - \frac{77}{110160} l^{25} + O\left(l^{33}\right) \tag{A82}$$

Using the above polynomial, the following equation is obtained:

$$\frac{d^2}{dl^2} sleaf_4(l) = -4l^7 + \frac{14}{9} l^{15} - \frac{385}{918} l^{23} + O\left(l^{31}\right) \tag{A83}$$

Using Eq. (A82), the following equation is obtained:

$$\begin{aligned} & -4 \cdot \left(sleaf_4(l)\right)^7 \\ &= -4 \cdot \left(l - \frac{1}{18} l^9 + \frac{7}{1224} l^{17} - \frac{77}{110160} l^{25} + O\left(l^{33}\right)\right)^7 \\ &= -4l^7 + \frac{14}{9} l^{15} - \frac{385}{918} l^{23} + O\left(l^{31}\right) \end{aligned} \tag{A84}$$

We can find that the leaf function $sleaf_4(l)$ is satisfied with Eq. (1).

Next, the Taylor expansion is applied to the leaf function: $cleaf_4(l)$. The first derivative of the leaf function $cleaf_4(l)$ is obtained as follows:

$$\frac{d}{dl} cleaf_4(l) = -\sqrt{1 - cleaf_4^8(l)} \tag{A85}$$

The second derivative of the leaf function $cleaf_4(l)$ is obtained as follows:

$$\frac{d^2}{dl^2} cleaf_4(l) = -4 \cdot cleaf_4^7(l) \tag{A86}$$

The third derivative of the leaf function $cleaf_4(l)$ is obtained as follows:

$$\frac{d^3}{dl^3} cleaf_4(l) = 28 \cdot cleaf_4^6(l) \cdot \sqrt{1 - cleaf_4^8(l)} \tag{A87}$$

The fourth derivative of the leaf function $cleaf_4(l)$ is obtained as follows:

$$\frac{d^4}{dl^4} cleaf_4(l) = 56 \cdot cleaf_4^5(l) \cdot \left(-3 + 5 cleaf_4^8(l)\right) \tag{A88}$$

The fifth derivative of the leaf function *cleaf₄(l)* is obtained as follows:

$$\frac{d^5}{dl^5} cleaf_4(l) = 280 cleaf_4^4(l)\left(3 - 13 cleaf_4^8(l)\right)\sqrt{1 - cleaf_4^8(l)} \tag{A89}$$

The sixth derivative of the leaf function *cleaf₄(l)* is obtained as follows:

$$\frac{d^6}{dl^6} cleaf_4(l) = -1120 cleaf_4^3(l)\left(3 - 45 cleaf_4^8(l) + 52 cleaf_4^{16}(l)\right) \tag{A90}$$

The seventh derivative of the leaf function *cleaf₄(l)* is obtained as follows:

$$\frac{d^7}{dl^7} cleaf_4(l) = 1120 cleaf_4^2(l)\sqrt{1 - cleaf_4^8(l)} \cdot \left(9 - 495 cleaf_4^8(l) + 988 cleaf_4^{16}(l)\right) \tag{A91}$$

The eighth derivative of the leaf function *cleaf₄(l)* is obtained as follows:

$$\frac{d^8}{dl^8} cleaf_4(l) = 2240 cleaf_4(l) \cdot \left(-9 + 2502 cleaf_4^8(l) - 12357 cleaf_4^{16}(l) + 10868 cleaf_4^{24}(l)\right) \tag{A92}$$

The ninth derivative of the leaf function *cleaf₄(l)* is obtained as follows:

$$\frac{d^9}{dl^9} cleaf_4(l) = -2240\sqrt{1 - cleaf_4^8(l)} \cdot \left(-9 + 22518 cleaf_4^8(l) - 210069 cleaf_4^{16}(l) + 271700 cleaf_4^{24}(l)\right) \tag{A93}$$

The tenth derivative of the leaf function *cleaf₄(l)* is obtained as follows:

$$\frac{d^{10}}{dl^{10}} cleaf_4(l) = -313600 cleaf_4^7(l) \cdot \left(-1287 + 25938 cleaf_4^8(l) - 76587 cleaf_4^{16}(l) + 54340 cleaf_4^{24}(l)\right) \tag{A94}$$

The eleventh derivative of the leaf function *cleaf₄(l)* is obtained as follows:

$$\frac{d^{11}}{dl^{11}} cleaf_4(l) = 313600 cleaf_4^6(l)\sqrt{1 - cleaf_4^8(l)} \cdot \left(-9009 + 389070 cleaf_4^8(l) - 1761501 cleaf_4^{16}(l) + 1684540 cleaf_4^{24}(l)\right) \tag{A95}$$

The twelfth derivative of the leaf function *cleaf₄(l)* is obtained as follows:

$$\frac{d^{12}}{dl^{12}} cleaf_4(l) = 627200 cleaf_4^5(l) \cdot \left[27027 + 17 cleaf_4^8(l)\left\{-162855 + 13 cleaf_4^8(l)\left(103521 - 217953 cleaf_4^8(l) + 129580 cleaf_4^{16}(l)\right)\right\}\right] \tag{A96}$$

It continues in the same way below. Using the Taylor expansion, the polynomial is obtained as follows:

$$\begin{aligned} cleaf_4(l) &= 1 - \frac{4}{2!}l^2 + \frac{112}{4!}l^4 - \frac{11200}{6!}l^6 + \frac{2248960}{8!}l^8 \\ &- \frac{753894400}{10!}l^{10} + \frac{3802086400\ 00}{12!}l^{12} + O\left(l^{14}\right) \\ &= 1 - 2l^2 + \frac{14}{3}l^4 - \frac{140}{9}l^6 + \frac{502}{9}l^8 \\ &- \frac{16828}{81}l^{10} + \frac{2121700}{2673}l^{12} + O\left(l^{14}\right) \end{aligned} \tag{A97}$$

Using the above polynomial, the following equation is obtained:

$$\begin{aligned} \frac{d^2}{dl^2} cleaf_4(l) &= -4 + 56l^2 - \frac{1400}{3}l^4 + \frac{28112}{9}l^6 \\ &- \frac{168280}{9}l^8 + \frac{8486800}{81}l^{10} + O\left(l^{12}\right) \end{aligned} \tag{A98}$$

Using Eq. (A97), the following equation is obtained:

$$\begin{aligned} &-4 \cdot \left(cleaf_4(l)\right)^7 \\ &= -3 \cdot \left(1 - 2l^2 + \frac{14}{3}l^4 - \frac{140}{9}l^6 + \frac{502}{9}l^8 - \frac{16828}{81}l^{10} + \frac{2121700}{2673}l^{12} + O\left(l^{14}\right)\right)^7 \\ &= -3 + \frac{45}{2}l^2 - \frac{765}{8}l^4 + \frac{5085}{16}l^6 - \frac{836055}{896}l^8 + \frac{4564989}{1792}l^{10} + O\left(l^{12}\right) \end{aligned} \tag{A99}$$

We can find that the leaf function *cleaf₄(l)* is satisfied with Eq. (1).

Next, in the case of $n=5$, the Taylor expansion is applied to the leaf function. The first derivative of the leaf function $sleaf_5(l)$ is obtained as follows:

$$\frac{d}{dl} sleaf_5(l) = \sqrt{1 - sleaf_5^{10}(l)} = cleaf_5^5(\bar{l}) \tag{A100}$$

The second derivative of the leaf function $sleaf_5(l)$ is obtained as follows:

$$\frac{d^2}{dl^2} sleaf_5(l) = -5 \cdot sleaf_5^9(l) \tag{A101}$$

The third derivative of the leaf function $sleaf_5(l)$ is obtained as follows:

$$\frac{d^3}{dl^3} sleaf_5(l) = -45 \cdot sleaf_5^8(l) \cdot \sqrt{1 - sleaf_5^{10}(l)} \tag{A102}$$

The fourth derivative of the leaf function $sleaf_5(l)$ is obtained as follows:

$$\frac{d^4}{dl^4} sleaf_5(l) = 45 \cdot sleaf_5^7(l) \cdot \left(-8 + 13 sleaf_5^{10}(l)\right) \tag{A103}$$

The fifth derivative of the leaf function $sleaf_5(l)$ is obtained as follows:

$$\frac{d^5}{dl^5} sleaf_5(l) = 45 sleaf_5^6(l)\left(-56 + 221 sleaf_5^{10}(l)\right)\sqrt{1 - sleaf_5^{10}(l)} \tag{A104}$$

The sixth derivative of the leaf function $sleaf_5(l)$ is obtained as follows:

$$\begin{aligned} &\frac{d^6}{dl^6} sleaf_5(l) \\ &= -135 sleaf_5^5(l)\left(112 - 1384 sleaf_5^{10}(l) + 1547 sleaf_5^{20}(l)\right) \end{aligned} \tag{A105}$$

The seventh derivative of the leaf function $sleaf_5(l)$ is obtained as follows:

$$\begin{aligned} &\frac{d^7}{dl^7} sleaf_5(l) = -675 sleaf_5^4(l)\sqrt{1 - sleaf_5^{10}(l)} \\ &\cdot\left(112 - 4152 sleaf_5^{10}(l) + 7735 sleaf_5^{20}(l)\right) \end{aligned} \tag{A106}$$

The eighth derivative of the leaf function $sleaf_5(l)$ is obtained as follows:

$$\begin{aligned} &\frac{d^8}{dl^8} sleaf_5(l) = 675 sleaf_5^3(l) \\ &\cdot\left(-448 + 59136 sleaf_5^{10}(l) - 264528 sleaf_5^{20}(l) + 224315 sleaf_5^{30}(l)\right) \end{aligned} \tag{A107}$$

The ninth derivative of the leaf function $sleaf_5(l)$ is obtained as follows:

$$\begin{aligned} &\frac{d^9}{dl^9} sleaf_5(l) = 2025 sleaf_5^2(l)\sqrt{1 - sleaf_5^{10}(l)} \\ &\cdot\left(-448 + 256256 sleaf_5^{10}(l) - 2028048 sleaf_5^{20}(l) + 2467465 sleaf_5^{30}(l)\right) \end{aligned} \tag{A108}$$

The tenth derivative of the leaf function $sleaf_5(l)$ is obtained as follows:

$$\begin{aligned} &\frac{d^{10}}{dl^{10}} sleaf_5(l) = -2025 sleaf_5(l) \cdot \left(896 - 3078208 sleaf_5^{10}(l)\right. \\ &\left. + 48973408 sleaf_5^{20}(l) - 133716176 sleaf_5^{30}(l) + 91296205 sleaf_5^{40}(l)\right) \end{aligned} \tag{A109}$$

The eleventh derivative of the leaf function $sleaf_5(l)$ is obtained as follows:

$$\begin{aligned} &\frac{d^{11}}{dl^{11}} sleaf_5(l) = -2025\sqrt{1 - sleaf_5^{10}(l)} \\ &\cdot\left\{896 + 11 sleaf_5^{10}(l)\left(-3078208 + 93494688 sleaf_5^{10}(l) - 376836496 sleaf_5^{20}(l) + 340285855 sleaf_5^{30}(l)\right)\right\} \end{aligned} \tag{A110}$$

It continues in the same way below. Using the Taylor expansion, the polynomial is obtained as follows:

$$\begin{aligned} sleaf_5(l) &= l - \frac{1}{22} l^{11} + \frac{3}{616} l^{21} - \frac{267}{420112} l^{31} \\ &+ \frac{136545}{1515764096} l^{41} + O\left(l^{51}\right) \end{aligned} \tag{A111}$$

Using the above polynomial, the following equation is obtained:

$$\frac{d^2}{dl^2} sleaf_5(l) = -5l^9 + \frac{45}{22}l^{19} - \frac{4005}{6776}l^{29} + \frac{682725}{4621232}l^{39} + O(l^{49}) \quad \text{(A112)}$$

Using Eq. (A111), the following equation is obtained:

$$\begin{aligned} &-5 \cdot (sleaf_5(l))^9 \\ &= -5 \cdot \left(l - \frac{1}{22}l^{11} + \frac{3}{616}l^{21} - \frac{267}{420112}l^{31} + \frac{136545}{1515764096}l^{41} + O(l^{51}) \right)^9 \\ &= -5l^9 + \frac{45}{22}l^{19} - \frac{4005}{6776}l^{29} + \frac{682725}{4621232}l^{39} + O(l^{49}) \end{aligned} \quad \text{(A113)}$$

We can find that the leaf function *sleaf₅(l)* is satisfied with Eq. (1).

Next, the Taylor expansion is applied to the leaf function: $cleaf_5(l)$. The first derivative of the leaf function $cleaf_5(l)$ is obtained as follows:

$$\frac{d}{dl} cleaf_5(l) = -\sqrt{1 - cleaf_5^{10}(l)} \quad \text{(A114)}$$

The second derivative of the leaf function $cleaf_5(l)$ is obtained as follows:

$$\frac{d^2}{dl^2} cleaf_5(l) = -5 \cdot cleaf_5^9(l) \quad \text{(A115)}$$

The third derivative of the leaf function $cleaf_5(l)$ is obtained as follows:

$$\frac{d^3}{dl^3} cleaf_5(l) = 45 \cdot cleaf_5^{10}(l) \cdot \sqrt{1 - cleaf_5^{10}(l)} \quad \text{(A116)}$$

The fourth derivative of the leaf function $cleaf_5(l)$ is obtained as follows:

$$\frac{d^4}{dl^4} cleaf_5(l) = 45 \cdot cleaf_5^7(l) \cdot \left(-8 + 13 cleaf_5^{10}(l)\right) \quad \text{(A117)}$$

The fifth derivative of the leaf function $cleaf_5(l)$ is obtained as follows:

$$\begin{aligned} &\frac{d^5}{dl^5} cleaf_5(l) \\ &= 45 cleaf_5^6(l)\left(56 - 221 cleaf_5^{10}(l)\right)\sqrt{1 - cleaf_5^{10}(l)} \end{aligned} \quad \text{(A118)}$$

The sixth derivative of the leaf function $cleaf_5(l)$ is obtained as follows:

$$\begin{aligned} &\frac{d^6}{dl^6} cleaf_5(l) \\ &= -135 cleaf_5^5(l)\left(112 - 1384 cleaf_5^{10}(l) + 1547 cleaf_5^{20}(l)\right) \end{aligned} \quad \text{(A119)}$$

The seventh derivative of the leaf function $cleaf_5(l)$ is obtained as follows:

$$\begin{aligned} &\frac{d^7}{dl^7} cleaf_5(l) = 675 cleaf_5^4(l)\sqrt{1 - cleaf_5^{10}(l)} \\ &\cdot \left(112 - 4152 cleaf_5^{10}(l) + 7735 cleaf_5^{20}(l)\right) \end{aligned} \quad \text{(A120)}$$

The eighth derivative of the leaf function $cleaf_5(l)$ is obtained as follows:

$$\begin{aligned} &\frac{d^8}{dl^8} cleaf_5(l) = 675 cleaf_5^3(l) \\ &\cdot \left(-448 + 59136 cleaf_5^{10}(l) - 264528 cleaf_5^{20}(l) + 224315 cleaf_5^{30}(l)\right) \end{aligned} \quad \text{(A121)}$$

The ninth derivative of the leaf function $cleaf_5(l)$ is obtained as follows:

$$\begin{aligned} &\frac{d^9}{dl^9} cleaf_5(l) = -2025 cleaf_5^2(l)\sqrt{1 - cleaf_5^{10}(l)} \\ &\cdot \left(-448 + 256256 cleaf_5^{10}(l) - 2028048 cleaf_5^{20}(l) + 2467465 cleaf_5^{30}(l)\right) \end{aligned} \quad \text{(A122)}$$

The tenth derivative of the leaf function $cleaf_5(l)$ is obtained as follows:

$$\begin{aligned} &\frac{d^{10}}{dl^{10}} cleaf_5(l) = -2025 sleaf_5(l) \cdot \left(896 - 3078208 cleaf_5^{10}(l)\right. \\ &\left. + 48973408 cleaf_5^{20}(l) - 133716176 cleaf_5^{30}(l) + 91296205 cleaf_5^{40}(l)\right) \end{aligned} \quad \text{(A123)}$$

The eleventh derivative of the leaf function $cleaf_5(l)$ is obtained as follows:

$$\begin{aligned} &\frac{d^{11}}{dl^{11}} cleaf_5(l) = 2025\sqrt{1 - cleaf_5^{10}(l)} \\ &\cdot \left\{896 + 11 cleaf_5^{10}(l)\left(-3078208 + 93494688 cleaf_5^{10}(l) - 376836496 cleaf_5^{20}(l) + 340285855 cleaf_5^{30}(l)\right)\right\} \end{aligned} \quad \text{(A124)}$$

It continues in the same way below. Using the Taylor expansion, the polynomial is obtained as follows:

$$cleaf_5(l) = 1 - \frac{5}{2}l^2 + \frac{75}{8}l^4 - \frac{825}{16}l^6 + \frac{277125}{896}l^8 - \frac{3476125}{1792}l^{10} + O(l^{12}) \tag{A125}$$

Using the above polynomial, the following equation is obtained:

$$\frac{d^2}{dl^2}cleaf_5(l) = -5 + \frac{225}{2}l^2 - \frac{12375}{8}l^4 + \frac{277125}{16}l^6 - \frac{156425625}{896}l^8 + O(l^{10}) \tag{A126}$$

Using Eq.(A125), the following equation is obtained:

$$\begin{aligned} & -5\cdot(cleaf_5(l))^9 \\ &= -5\cdot\left(1 - \frac{5}{2}l^2 + \frac{75}{8}l^4 - \frac{825}{16}l^6 + \frac{277125}{896}l^8 - \frac{3476125}{1792}l^{10} + O(l^{12})\right)^9 \\ &= -5 + \frac{225}{2}l^2 - \frac{12375}{8}l^4 + \frac{277125}{16}l^6 - \frac{156425625}{896}l^8 + O(l^{10}) \end{aligned} \tag{A127}$$

We can find that the leaf function *cleaf₅(l)* is satisfied with Eq. (1).

Appendix B

We prove that Eq. (67) is satisfied with respect to the arbitrary variable *l*. The following equation is considered.

$$x^2 + y^2 + 2\cdot x^2\cdot y^2 = 1 \tag{B1}$$

By solving the above equation with respect to the variable *y*, the following equation is obtained:

$$y = \sqrt{\frac{1-x^2}{1+2x^2}} \tag{B2}$$

The above equation is differentiated with respect to the variable *x*.

$$\frac{dy}{dx} = -\frac{3x}{(1+2x^2)^{\frac{3}{2}}\sqrt{1-x^2}} \tag{B3}$$

By multiplying $\sqrt{1+x^2+x^4}$ in both the numerator and denominator, the following equation is obtained.

$$\begin{aligned} \frac{dy}{dx} &= -\frac{3x\sqrt{1+x^2+x^4}}{(1+2x^2)^{\frac{3}{2}}\sqrt{1-x^2}\sqrt{1+x^2+x^4}} \\ &= -\frac{3x\sqrt{1+x^2+x^4}}{(1+2x^2)^{\frac{3}{2}}\sqrt{1-x^6}} \end{aligned} \tag{B4}$$

Using Eq. (B2), the following equation is obtained.

$$\frac{1}{\sqrt{1-y^6}} = \frac{1}{\sqrt{1-\left(\sqrt{\frac{1-x^2}{1+2x^2}}\right)^6}} = \frac{(1+2x^2)^{\frac{3}{2}}}{3x\sqrt{1+x^2+x^4}} \tag{B5}$$

By Eq.(B4) and Eq.(B5), the ODE is obtained as follows:

$$\frac{dy}{\sqrt{1-y^6}} + \frac{dx}{\sqrt{1-x^6}} = 0 \tag{B6}$$

The following equation is defined.

$$x = cleaf_3(l) \tag{B7}$$

$$y = sleaf_3(l) \tag{B8}$$

Eq. (B7) is differentiated with respect to the variable *l*.

$$\frac{dx}{dl} = -\sqrt{1-cleaf_3^6(l)} = -\sqrt{1-x^6} \tag{B9}$$

Eq. (B8) is differentiated with respect to the variable *l*.

$$\frac{dy}{dl} = \sqrt{1-sleaf_3^6(l)} = \sqrt{1-y^6} \tag{B10}$$

By substituting both Eq. (B9) and Eq. (B10) in Eq. (B6), the following equation is obtained.

$$\frac{dy}{\sqrt{1-y^6}}+\frac{dx}{\sqrt{1-x^6}}=\frac{1}{\sqrt{1-y^6}}\frac{dy}{dl}dl+\frac{1}{\sqrt{1-x^6}}\frac{dx}{dl}dl$$
$$=\frac{\sqrt{1-y^6}\cdot dl}{\sqrt{1-y^6}}-\frac{\sqrt{1-x^6}\cdot dl}{\sqrt{1-x^6}}=0 \quad \text{(B11)}$$

By satisfying Eq.(B6), the Eq.(B7) and Eq.(B8) are satisfied with Eq. (B1).

$$x^2+y^2+2\cdot x^2\cdot y^2$$
$$=(cleaf_3(l))^2+(sleaf_3(l))^2+2\cdot(cleaf_3(l))^2\cdot(sleaf_3(l))^2$$
$$=1 \quad \text{(B12)}$$

Appendix C

The Euler's formula is as follows:

$$e^{i\theta}=\cos(\theta)+i\sin(\theta) \quad \text{(C1)}$$

Using the Euler's formula, we can obtain the relation between the complex exponential function and the leaf function.

$$e^{i\cdot n\int_0^l(cleaf_n(t))^{n-1}dt}=\cos\left(n\int_0^l(cleaf_n(t))^{n-1}dt\right)$$
$$+i\sin\left(n\int_0^l(cleaf_n(t))^{n-1}dt\right) \quad \text{(C2)}$$
$$=\cos\left(n\int_0^l(cleaf_n(t))^{n-1}dt\right)$$
$$\pm i\sqrt{1-\left\{\cos\left(n\int_0^l(cleaf_n(t))^{n-1}dt\right)\right\}^2}$$

Using the Eq. (58), the above equation is as follows:

$$e^{i\cdot n\int_0^l(cleaf_n(t))^{n-1}dt}=(cleaf_n(l))^n\pm i\sqrt{1-(cleaf_n(l))^{2n}} \quad \text{(C3)}$$

By taking the log_e of both sides of the above equation, we can make another equation:

$$\int_0^l(cleaf_n(t))^{n-1}dt=-\frac{i}{n}\log_e\left\{(cleaf_n(l))^n\pm i\sqrt{1-(cleaf_n(l))^{2n}}\right\} \quad \text{(C4)}$$

In the same way, the relation between the leaf function: $sleaf_n(l)$ and the complex exponential function can be derived:

$$e^{i\cdot n\int_0^l(sleaf_n(t))^{n-1}dt}=\pm\sqrt{1-(sleaf_n(l))^{2n}}+i(sleaf_n(l))^n \quad \text{(C5)}$$

By taking the log_e of both sides of the above equation, we can make another equation:

$$\int_0^l(sleaf_n(t))^{n-1}dt=-\frac{i}{n}\log_e\left\{\pm\sqrt{1-(sleaf_n(l))^{2n}}+i(sleaf_n(l))^n\right\} \quad \text{(C6)}$$

Appendix D

In the leaf functions $sleaf_n(l)$ and $cleaf_n(l)$ *(n=1,2,3,4,5,100)*, the numerical data with respect to variables *l* is summarized in Table 4-7.

Table 4 Numerical data of leaf function: *sleaf_n(l)* with respect to variables *l*

l	*r (=sleaf_n(l))*					
	n=1	*n=2*	*n=3*	*n=4*	*n=5*	*n=100*
0.0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0.1	0.099833	0.099999	0.099999	0.099999	0.100000	0.100000
0.2	0.198669	0.199968	0.199999	0.199999	0.200000	0.200000
0.3	0.295520	0.299757	0.299984	0.299998	0.300000	0.300000
0.4	0.389418	0.398978	0.399883	0.399985	0.399999	0.400000
0.5	0.479425	0.496891	0.499442	0.499891	0.499978	0.500000
0.6	0.564642	0.592307	0.598009	0.599441	0.599836	0.600000
0.7	0.644217	0.683522	0.694183	0.697771	0.699104	0.700000
0.8	0.717356	0.768313	0.785387	0.792669	0.796140	0.800000
0.9	0.783326	0.844009	0.867486	0.879382	0.886246	0.900000
1.0	0.841470	0.907683	0.934767	0.949545	0.958859	0.997779
1.1	0.891207	0.956432	0.980707	0.991987	0.997401	0.913828
1.2	0.932039	0.987748	0.999692	0.997357	0.988736	0.813828
1.3	0.963558	0.999878	0.989089	0.964307	0.936130	0.713828
1.4	0.985449	0.992115	0.950392	0.900534	0.855664	0.613828
1.5	0.997494	0.964914	0.888559	0.817333	0.762260	0.513828
1.6	0.999573	0.919815	0.810063	0.724051	0.664110	0.413828
1.7	0.991664	0.859192	0.720971	0.626352	0.564532	0.313828
1.8	0.973847	0.785891	0.625895	0.527010	0.464607	0.213828
1.9	0.946300	0.702864	0.527828	0.427158	0.364616	0.113828
2.0	0.909297	0.612857	0.428460	0.327182	0.264617	0.013828
2.1	0.863209	0.518203	0.328621	0.227185	0.164617	-0.086172
2.2	0.808496	0.420721	0.228648	0.127185	0.064617	-0.186172
2.3	0.745705	0.321711	0.128650	0.027185	-0.035383	-0.286172
2.4	0.675463	0.222003	0.028650	-0.072814	-0.135383	-0.386172
2.5	0.598472	0.122054	-0.071349	-0.172814	-0.235383	-0.486172
2.6	0.515501	0.022057	-0.171349	-0.272814	-0.335383	-0.586172
2.7	0.427379	-0.077942	-0.271341	-0.3728071	-0.435378	-0.686172
2.8	0.334988	-0.177924	-0.371279	-0.4727492	-0.535336	-0.786172
2.9	0.239249	-0.277776	-0.470980	-0.5724464	-0.635074	-0.886172
3.0	0.141120	-0.377172	-0.569934	-0.6712520	-0.733845	-0.986021
3.1	0.041580	-0.475459	-0.666997	-0.7674226	-0.829207	-0.927657
3.2	-0.058374	-0.571553	-0.759974	-0.8570272	-0.914706	-0.827657

Table 5 Numerical data of leaf function: $sleaf_n(l)$ with respect to variables l

l	$r\ (=sleaf_n(l))$					
	n=1	n=2	n=3	n=4	n=5	n=100
3.3	-0.157745	-0.663869	-0.845202	-0.932725	-0.977301	-0.727657
3.4	-0.255541	-0.750293	-0.917385	-0.983827	-0.999976	-0.627657
3.5	-0.350783	-0.828242	-0.970094	-0.999829	-0.974439	-0.527657
3.6	-0.442520	-0.894823	-0.997235	-0.976779	-0.909941	-0.427657
3.7	-0.529836	-0.947099	-0.995142	-0.920265	-0.823533	-0.327657
3.8	-0.611857	-0.982443	-0.964109	-0.841275	-0.727831	-0.227657
3.9	-0.687766	-0.998905	-0.908268	-0.750021	-0.628956	-0.127657
4	-0.756802	-0.995532	-0.833879	-0.653152	-0.529193	-0.027657
4.1	-0.818277	-0.972521	-0.747279	-0.554095	-0.42923	0.072343
4.2	-0.871575	-0.931190	-0.653544	-0.454324	-0.329234	0.172343
4.3	-0.916165	-0.873757	-0.556112	-0.354365	-0.229234	0.272343
4.4	-0.951602	-0.802997	-0.457002	-0.254370	-0.129234	0.372343
4.5	-0.977530	-0.721869	-0.357248	-0.154370	-0.029234	0.472343
4.6	-0.993691	-0.633184	-0.257295	-0.054370	0.070766	0.572343
4.7	-0.999923	-0.539380	-0.157301	0.045629	0.170766	0.672343
4.8	-0.996164	-0.442392	-0.057301	0.145629	0.270766	0.772343
4.9	-0.982452	-0.343633	0.042698	0.245629	0.370765	0.872343
5	-0.958924	-0.244028	0.142698	0.345625	0.470754	0.972334
5.1	-0.925814	-0.144108	0.242695	0.445591	0.570671	0.941486
5.2	-0.883454	-0.044115	0.342659	0.545391	0.670205	0.841486
5.3	-0.832267	0.055884	0.442460	0.644550	0.768194	0.741486
5.4	-0.772764	0.155875	0.541710	0.741710	0.861104	0.641486
5.5	-0.705540	0.255775	0.639485	0.833665	0.940351	0.541486
5.6	-0.631266	0.355314	0.733936	0.914090	0.990663	0.441486
5.7	-0.550685	0.453922	0.821856	0.973033	0.996323	0.341486
5.8	-0.464602	0.550618	0.898403	0.999355	0.955193	0.241486
5.9	-0.373876	0.643931	0.957365	0.986746	0.881066	0.141486
6	-0.279415	0.731861	0.992353	0.938343	0.790303	0.041486
6.1	-0.182162	0.811921	0.998793	0.864333	0.693038	-0.0585140
6.2	-0.083089	0.881266	0.975775	0.775596	0.593705	-0.158514
6.3	0.016813	0.936940	0.926454	0.679800	0.493832	-0.258514

Table 6 Numerical data of leaf function: $cleaf_n(l)$ with respect to variables l

l	*r (=cleaf_n(l))*					
	n=1	*n=2*	*n=3*	*n=4*	*n=5*	*n=100*
0.0	1.000000	1.000000	1.000000	1.000000	1.00000	1.000000
0.1	0.995004	0.990049	0.985184	0.980451	0.975888	0.906914
0.2	0.980066	0.960781	0.942809	0.926595	0.912333	0.806914
0.3	0.955336	0.913842	0.878183	0.849205	0.826374	0.706914
0.4	0.921060	0.851676	0.797825	0.758746	0.730839	0.606914
0.5	0.877582	0.777159	0.707632	0.662211	0.632015	0.506914
0.6	0.825335	0.693234	0.611978	0.563274	0.532264	0.406914
0.7	0.764842	0.602609	0.513646	0.463537	0.432304	0.306914
0.8	0.696706	0.507563	0.414175	0.363586	0.332308	0.206914
0.9	0.621609	0.409858	0.314303	0.263592	0.232308	0.106914
1.0	0.540302	0.310737	0.214323	0.163592	0.132308	0.006914
1.1	0.453596	0.210986	0.114325	0.063592	0.032308	-0.093086
1.2	0.362357	0.111027	0.014325	-0.036407	-0.067692	-0.193086
1.3	0.267498	0.011028	-0.085674	-0.136407	-0.167692	-0.293086
1.4	0.169967	-0.088970	-0.185674	-0.236407	-0.267692	-0.393086
1.5	0.070737	-0.188947	-0.285663	-0.336404	-0.367691	-0.493086
1.6	-0.029199	-0.288769	-0.385584	-0.436375	-0.467681	-0.593086
1.7	-0.128844	-0.388082	-0.485219	-0.536203	-0.567602	-0.693086
1.8	-0.227202	-0.486189	-0.583992	-0.635458	-0.667158	-0.793086
1.9	-0.323289	-0.581954	-0.680635	-0.732900	-0.765230	-0.893086
2.0	-0.416146	-0.673733	-0.772765	-0.825544	-0.858391	-0.992488
2.1	-0.504846	-0.759356	-0.856486	-0.907398	-0.938254	-0.920743
2.2	-0.588501	-0.836197	-0.926286	-0.968806	-0.989722	-0.820743
2.3	-0.666276	-0.901342	-0.975673	-0.998524	-0.996884	-0.720743
2.4	-0.737393	-0.951871	-0.998769	-0.989524	-0.957042	-0.620743
2.5	-0.801143	-0.985211	-0.992412	-0.944057	-0.883663	-0.520743
2.6	-0.856888	-0.999513	-0.957498	-0.871923	-0.793224	-0.420743
2.7	-0.904072	-0.993943	-0.898594	-0.784163	-0.696072	-0.320743
2.8	-0.942222	-0.968828	-0.822087	-0.688798	-0.596770	-0.220743
2.9	-0.970958	-0.925599	-0.734190	-0.590291	-0.496905	-0.120743
3.0	-0.989992	-0.866554	-0.639752	-0.490685	-0.396924	-0.020743
3.1	-0.999135	-0.794505	-0.541984	-0.390765	-0.296926	0.079257
3.2	-0.998294	-0.712411	-0.442737	-0.290776	-0.196926	0.179257

Table 7 Numerical data of leaf function: $cleaf_n(l)$ with respect to variables l

l	r $(=cleaf_n(l))$					
	n=1	n=2	n=3	n=4	n=5	n=100
3.3	-0.987479	-0.623050	-0.342936	-0.190777	-0.096926	0.279257
3.4	-0.966798	-0.528809	-0.242972	-0.090777	0.003074	0.379257
3.5	-0.936456	-0.431567	-0.142975	0.009222	0.103074	0.479257
3.6	-0.896758	-0.332676	-0.042975	0.109222	0.203074	0.579257
3.7	-0.848100	-0.233017	0.057024	0.209222	0.303074	0.679257
3.8	-0.790967	-0.133082	0.157023	0.309220	0.403072	0.779257
3.9	-0.725932	-0.033086	0.257018	0.409204	0.503051	0.879257
4	-0.653643	0.066913	0.356971	0.509094	0.602900	0.979220
4.1	-0.574823	0.166900	0.456726	0.608581	0.702134	0.934572
4.2	-0.490260	0.266778	0.555838	0.706716	0.799050	0.834572
4.3	-0.400799	0.366249	0.653277	0.801108	0.888812	0.734572
4.4	-0.307332	0.464703	0.747027	0.886703	0.960643	0.634572
4.5	-0.210795	0.561107	0.833653	0.954795	0.997871	0.534572
4.6	-0.112152	0.653934	0.908084	0.994128	0.987709	0.434572
4.7	-0.012388	0.741126	0.963986	0.995855	0.933980	0.334572
4.8	0.087498	0.820148	0.995094	0.959545	0.852927	0.234572
4.9	0.186512	0.888129	0.997271	0.893507	0.759288	0.134572
5	0.283662	0.942122	0.970207	0.809040	0.661061	0.034572
5.1	0.377977	0.979443	0.917561	0.715167	0.561463	-0.065428
5.2	0.468516	0.998055	0.845423	0.617234	0.461534	-0.165428
5.3	0.554374	0.996881	0.760223	0.517813	0.361542	-0.265428
5.4	0.634692	0.975990	0.667261	0.417941	0.261543	-0.365428
5.5	0.708669	0.936585	0.570206	0.317961	0.161543	-0.465428
5.6	0.775565	0.880799	0.471256	0.217962	0.061543	-0.565428
5.7	0.834712	0.811364	0.371556	0.117962	-0.038457	-0.665428
5.8	0.885519	0.731235	0.271618	0.017962	-0.138457	-0.765428
5.9	0.927478	0.643256	0.171626	-0.082037	-0.238457	-0.865428
6	0.960170	0.549912	0.071626	-0.182037	-0.338457	-0.965426
6.1	0.983268	0.453197	-0.028373	-0.282036	-0.438452	-0.948401
6.2	0.996542	0.354579	-0.128373	-0.382027	-0.538407	-0.848401
6.3	0.999858	0.255035	-0.228371	-0.481959	-0.638131	-0.748401