

Special Function: Hyperbolic Leaf Function $r=sleafh_n(l)$ (First Report)

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Summary

In the previous study, the leaf functions $sleaf_n(l)$ and $cleaf_n(l)$ are defined. These functions satisfy the ordinary differential equation (ODE) as follows:

$$\frac{d^2 r(l)}{dl^2} = -n \cdot r(l)^{2n-1} \quad n = 1, 2, 3, \dots$$

The variable $r(l)$ consists of a parameter l . The parameter n represents the natural number. The number is defined as the basis. Graphs of these functions are obtained by solving the above ODE. We find that these functions have the periodicity through these graphs. The curves and periodicity of the leaf functions are different from that of the trigonometric functions.

In this study, we discuss the following ODE:

$$\frac{d^2 r(l)}{dl^2} = n \cdot r(l)^{2n-1} \quad n = 1, 2, 3, \dots$$

In the right side of the above equation, the sign is replaced from “-“ to “+”. The initial conditions are defined as follows:

$$\begin{aligned} r(0) &= 0 \\ \frac{dr(0)}{dl} &= 1 \end{aligned}$$

The hyperbolic leaf function $r=sleafh_n(l)$ is defined as the solution of the above ODE with the initial conditions. In the case of the basis $n=1$, the hyperbolic leaf function $sleafh_1(l)$ represents the hyperbolic function $\sinh(l)$. With respect to an arbitrary basis n , the leaf hyperbolic function $sleafh_n(l)$ is closely related to the leaf function $sleaf_n(l)$.

Keywords : Leaf function, Leaf curve, Jacobi elliptic functions, Ordinary differential equation (ODE), Trigonometric function, Hyperbolic function, Square root of polynomial, Elliptic integral

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1. Introduction

To describe a natural phenomenon using a mathematical model, various ordinary differential equations (ODEs) are applied. These ODEs consist of unknown functions and their derivative function. Some solutions to the equation can be described as the elementary function, such as the trigonometric function and the exponential function, etc. On the other hand, various solutions of the ODE almost cannot be derived from the elementary function. Therefore, these solutions are typically computed using numerical analysis approach. The numerical error in this approach causes serious problems. The explanation on the process to derive the exact solution of ODEs is a more meaningful problem.

In some ODEs where exact solutions become unclear, the computational results by numerical analysis approach show properties of periodicity. An unknown function is raised to the $2n-1$ power. The parameter n represents the natural number, where the negative sign “-” is added. The function is equal to the second derivative of the unknown function. This equation is a case of an ODE with properties of periodicity. In the case of $n=1$, the unknown function represents the trigonometric function. In the case of $n=2$, the unknown function represents the elliptic function. In the case of $n=3$, to the best of our knowledge, the unknown function is unclear and has never been published. The unknown function is satisfied with the simple ODE. On the other hand, we can observe the periodicity with respect to the unknown function, which appears to be an important primitive function. Therefore, in the previous study, the unknown function is defined as the leaf function, which is discussed in the present study.

In the case of $n=1$, the unknown function represents the trigonometric function. As an analogous function with respect to the trigonometric function, the hyperbolic function exists. The hyperbolic function $\sinh(l)$ is differentiated with respect to the variable l .

$$\frac{d}{dl} \sinh(l) = \sqrt{1 + (\sinh(l))^2} \quad (1)$$

Eq. (1) is differentiated with respect to the variable l .

$$\frac{d^2}{dl^2} \sinh(l) = \frac{2 \sinh(l) \sqrt{1 + (\sinh(l))^2}}{2 \sqrt{1 + (\sinh(l))^2}} = \sinh(l) \quad (2)$$

By the second derivative of the hyperbolic function $\sinh(l)$, the function is returned to the original function $\sinh(l)$. The hyperbolic function r is satisfied with the following ODE:

$$\frac{d^2 r}{dl^2} = r \quad (3)$$

In previous study [1][2], the leaf function of the basis $n=1$ (the trigonometric function) is satisfied with the ODE as follows:

$$\frac{d^2 r}{dl^2} = -r \quad (4)$$

Comparing Eq. (3) with Eq. (4), the positive sign “+” in front of the variable r in Eq. (3) is replaced by the negative sign “-” as shown in Eq. (4). From these results, a hypothesis is considered. The leaf function is satisfied with the ODE as follows:

$$\frac{d^2 r}{dl^2} = -n \cdot r^{2n-1} \quad n = 1, 2, 3, \dots \quad (5)$$

With respect to arbitrary n , the ODE pair of Eq. (5) is assumed as follows:

$$\frac{d^2 r}{dl^2} = n \cdot r^{2n-1} \quad n = 1, 2, 3, \dots \quad (6)$$

A function may be satisfied with Eq. (6), which may be closely related to the leaf function.

In this study, we present the special function called hyperbolic leaf function $\text{leaf}_n(l)$, which satisfies Eq. (6). Then, we discuss the relation between the hyperbolic leaf function and the leaf function.

2. Definition of Hyperbolic Leaf Function

In this section, Eq. (6) is discussed. By multiplying dr/dl to both sides of Eq. (6), the following equation is obtained:

$$\frac{dr}{dl} \frac{d^2 r}{dl^2} = nr^{2n-1} \frac{dr}{dl} \quad n = 1, 2, 3, \dots \quad (7)$$

By integrating both sides of Eq. (7), the following equation is obtained.

$$\frac{1}{2} \left(\frac{dr}{dl} \right)^2 = \frac{1}{2} r^{2n} + C \quad n = 1, 2, 3, \dots \quad (8)$$

The symbol C represents the constant of integration. The constant C is determined by the initial conditions as follows:

$$r(0) = 0 \quad (9)$$

$$\frac{dr(0)}{dl} = 1 \quad (10)$$

Therefore, it is obtained as follows:

$$C = \frac{1}{2} \quad (11)$$

Using the above results and Eq. (8), the following equation is obtained as follows:

$$\frac{dr}{dl} = \pm \sqrt{1 + r^{2n}} \quad (12)$$

In the above equation, the positive sign “+” is demanded by the initial condition $dr(0)/dl=1>0$. Therefore, the above equation is as follows:

$$\frac{dr}{dl} = \sqrt{1 + r^{2n}} \quad (13)$$

As the variable r is increased, the function $\sqrt{1 + r^{2n}}$ is monotonically increased. After the variables are separated, it is integrated from 0 to r .

$$\int_0^r \frac{1}{\sqrt{1 + t^{2n}}} dt = l \quad (14)$$

The inverse function that satisfies the above equation is defined as follows:

$$asleafh_n(r) = \int_0^r \frac{1}{\sqrt{1 + t^{2n}}} dt = l \quad (15)$$

In this study, the prefix “a” of the hyperbolic leaf function $sleafh_n(l)$ represents the inverse function. Using the above equation, it is obtained as follows:

$$r = sleafh_n(l) \quad (16)$$

In the case of the basis $n=1$, the following equation is obtained:

$$sleafh_1(l) = \sinh(l) \quad (17)$$

3. Maclaurin Series of Hyperbolic Leaf Function

In this section, the Maclaurin series is applied to the hyperbolic leaf function. In the case of $n=2$, the function $sleafh_2(l)$ is expanded as follows:

$$\begin{aligned} sleafh_2(l) &= sleafh_2(0) + \frac{1}{1!} \left(\frac{d}{dl} sleafh_2(0) \right) l \\ &+ \frac{1}{2!} \left(\frac{d^2}{dl^2} sleafh_2(0) \right) l^2 + \frac{1}{3!} \left(\frac{d^3}{dl^3} sleafh_2(0) \right) l^3 + \\ &\dots + \frac{1}{13!} \left(\frac{d^{13}}{dl^{13}} sleafh_2(0) \right) l^{13} + \dots \\ &= \frac{1}{1!} l + \frac{12}{5!} l^5 + \frac{3024}{9!} l^9 + \frac{4390848}{13!} l^{13} + O(l^{15}) \\ &= l + \frac{1}{10} l^5 + \frac{1}{120} l^9 + \frac{11}{15600} l^{13} + O(l^{17}) \end{aligned} \quad (18)$$

For detailed information, see Appendix A. The symbol O represents the Landau symbol (the big O notation). The symbol $O(l^{17})$ represents the order of the error.

$$\lim_{l \rightarrow 0} \frac{O(l^{17})}{l^{17}} = \frac{sleafh_2(l) - \left(l + \frac{1}{10} l^5 + \frac{1}{120} l^9 + \frac{11}{15600} l^{13} \right)}{l^{17}} = \frac{211}{3536000} \quad (19)$$

Next, in the case of $n=3$, the hyperbolic leaf function $sleafh_3(l)$ can be expanded by the Maclaurin series as follows:

$$\begin{aligned} sleafh_3(l) &= \frac{1}{1!} l + \frac{360}{7!} l^7 + \frac{42768000}{13!} l^{13} \\ &+ \frac{91085575680000}{19!} l^{19} + O(l^{25}) \\ &= l + \frac{1}{14} l^7 + \frac{5}{728} l^{13} + \frac{145}{193648} l^{19} + O(l^{25}) \end{aligned} \quad (20)$$

In the case of $n=4$, the hyperbolic leaf function $sleafh_4(l)$ can be expanded by the Maclaurin series as follows:

$$sleafh_4(l) = l + \frac{1}{18} l^9 + \frac{7}{1224} l^{17} + \frac{77}{110160} l^{25} + O(l^{33}) \quad (21)$$

In the case of $n=5$, the hyperbolic leaf function $sleafh_5(l)$ can be expanded by the Maclaurin series as follows:

$$sleafh_5(l) = l + \frac{1}{22}l^{11} + \frac{3}{616}l^{21} + \frac{267}{420112}l^{31} + \frac{136545}{1515764096}l^{41} + O(l^{51}) \quad (22)$$

4. Relation between Leaf Function $sleaf_n(l)$ and Hyperbolic Leaf Function $sleafh_n(l)$

Using complex number i , we discuss the relation between the leaf function $sleaf_n(l)$ and hyperbolic leaf function $sleafh_n(l)$. The complex variable $i \cdot l$ is substituted for the variable l in the Maclaurin series of the function $sleaf_n(l)$ (see Ref. [2]). The symbol i represents the imaginary number. In the case of basis: $n=1$, the function $sleaf_1(l)$ and $sleafh_1(l)$ represent the function $\sin(l)$ and $\sinh(l)$, respectively. Therefore, Eq. (23) is obtained as follows:

$$sleaf_1(i \cdot l) = i \cdot sleafh_1(l) \quad (\sin(i \cdot l) = i \cdot \sinh(l)) \quad (23)$$

In the case of basis: $n=2$, it is obtained as follows:

$$\begin{aligned} sleafh_2(i \cdot l) &= i \cdot l - \frac{1}{10}(i \cdot l)^5 + \frac{1}{120}(i \cdot l)^9 - \frac{11}{15600}(i \cdot l)^{13} + O((i \cdot l)^{17}) \\ &= i \cdot l - \frac{1}{10}i^5 \cdot l^5 + \frac{1}{120}i^9 \cdot l^9 - \frac{11}{15600}i^{13} \cdot l^{13} + O(i^{17} \cdot l^{17}) \\ &= i \cdot l - i \cdot \frac{1}{10}l^5 + i \cdot \frac{1}{120}l^9 - i \cdot \frac{11}{15600}l^{13} + i \cdot O(l^{17}) = i \cdot sleafh_2(l) \end{aligned} \quad (24)$$

The complex variable $i \cdot l$ is substituted for the variable l in the Maclaurin series of the function $sleafh_2(l)$.

$$\begin{aligned} sleafh_2(i \cdot l) &= i \cdot l + \frac{1}{10}(i \cdot l)^5 + \frac{1}{120}(i \cdot l)^9 + \frac{11}{15600}(i \cdot l)^{13} + O((i \cdot l)^{17}) \\ &= i \cdot l + \frac{1}{10}i^5 \cdot l^5 + \frac{1}{120}i^9 \cdot l^9 + \frac{11}{15600}i^{13} \cdot l^{13} + O(i^{17} \cdot l^{17}) \\ &= i \cdot l + i \cdot \frac{1}{10}l^5 + i \cdot \frac{1}{120}l^9 + i \cdot \frac{11}{15600}l^{13} + i \cdot O(l^{17}) = i \cdot sleafh_2(l) \end{aligned} \quad (25)$$

In the case of basis: $n=3$, it is obtained as follows:

$$\begin{aligned} sleafh_3(i \cdot l) &= i \cdot l - \frac{1}{14}i^7 \cdot l^7 + \frac{5}{728}i^{13} \cdot l^{13} - \frac{145}{193648}i^{19} \cdot l^{19} \\ &+ \frac{4663}{54221440}i^{25} \cdot l^{25} - \frac{3111273}{305917364480}i^{31} \cdot l^{31} \\ &+ \frac{77686677}{63386077920256}i^{37} \cdot l^{37} - O(i^{43} \cdot l^{43}) \\ &= i \cdot l + \frac{1}{14}i \cdot l^7 + \frac{5}{728}i \cdot l^{13} + \frac{145}{193648}i \cdot l^{19} \\ &+ \frac{4663}{54221440}i \cdot l^{25} + \frac{3111273}{305917364480}i \cdot l^{31} \\ &+ \frac{77686677}{63386077920256}i \cdot l^{37} + i \cdot O(l^{43}) = i \cdot sleafh_3(l) \end{aligned} \quad (26)$$

In the case of basis: $n=4$, it is obtained as follows:

$$\begin{aligned} sleafh_4(i \cdot l) &= i \cdot l - \frac{1}{18}i^9 \cdot l^9 + \frac{7}{1224}i^{17} \cdot l^{17} \\ &- \frac{77}{110160}i^{25} \cdot l^{25} + O(i^{33} \cdot l^{33}) \\ &= i \cdot l - \frac{1}{18}i \cdot l^9 + \frac{7}{1224}i \cdot l^{17} - \frac{77}{110160}i \cdot l^{25} \\ &+ i \cdot O(l^{33}) = i \cdot sleafh_4(l) \end{aligned} \quad (27)$$

The complex variable $i \cdot l$ is substituted for the variable l in the Maclaurin series of the function $sleafh_4(l)$.

$$\begin{aligned} sleafh_4(i \cdot l) &= i \cdot l + \frac{1}{18}i^9 \cdot l^9 + \frac{7}{1224}i^{17} \cdot l^{17} \\ &+ \frac{77}{110160}i^{25} \cdot l^{25} + O(i^{33} \cdot l^{33}) \\ &= i \cdot l + \frac{1}{18}i \cdot l^9 + \frac{7}{1224}i \cdot l^{17} + \frac{77}{110160}i \cdot l^{25} \\ &+ i \cdot O(l^{33}) = i \cdot sleafh_4(l) \end{aligned} \quad (28)$$

In the case of basis: $n=5$, it is obtained as follows:

$$\begin{aligned} sleafh_5(i \cdot l) &= i \cdot l - \frac{1}{22}i^{11} \cdot l^{11} + \frac{3}{616}i^{21} \cdot l^{21} \\ &- \frac{267}{420112}i^{31} \cdot l^{31} + \frac{136545}{1515764096}i^{41} \cdot l^{41} + O(i^{51} \cdot l^{51}) \\ &= i \cdot l + \frac{1}{22}i \cdot l^{11} + \frac{3}{616}i \cdot l^{21} + \frac{267}{420112}i \cdot l^{31} \\ &+ \frac{136545}{1515764096}i \cdot l^{41} + i \cdot O(l^{51}) = i \cdot sleafh_5(l) \end{aligned} \quad (29)$$

Based on the above results in the case of odd number n ($n=2m-1$ ($m=1, 2, 3, \dots$)), the following equation can be predicted.

$$sleaf_{2m-1}(i \cdot l) = i \cdot sleafh_{2m-1}(l) \quad (m=1,2,3,\dots) \quad (30)$$

Based on the above results in the case of even number n ($n=2m$ ($m=1, 2, 3, \dots$)), the following equation can be predicted.

$$sleaf_{2m}(i \cdot l) = i \cdot sleaf_{2m}(l) \quad (m = 1, 2, 3, \dots) \quad (31)$$

$$sleafh_{2m}(i \cdot l) = i \cdot sleafh_{2m}(l) \quad (m = 1, 2, 3, \dots) \quad (32)$$

5. Relation between Leaf Function $sleaf_2(l)$ and Hyperbolic Leaf Function $sleafh_2(l)$

In the case of the basis $n=2$, the relation between the leaf function $sleaf_2(l)$ and the hyperbolic leaf function $sleafh_2(l)$ is derived as follows:

$$(sleaf_2(\sqrt{2} \cdot l))^2 = \frac{2(sleafh_2(l))^2}{1 + (sleafh_2(l))^4} \quad (33)$$

For detailed information, see Appendix B. The above equation also can be described as follows:

$$(sleafh_2(l))^2 = \frac{1 \pm \sqrt{1 - (sleaf_2(\sqrt{2} \cdot l))^4}}{(sleaf_2(\sqrt{2} \cdot l))^2} \quad (34)$$

Using Eq. (66) in Ref. [2], the relation between the function $sleaf_2(l)$ and $sleafh_2(l)$ can also be described as follows:

$$sleaf_2(\sqrt{2}l) = \frac{1 - (sleafh_2(l))^2}{1 + (sleafh_2(l))^2} \quad (35)$$

6. Graph of Hyperbolic Leaf Function $sleafh_n(l)$

The hyperbolic leaf function $sleafh_n(l)$ is shown in Fig. 1. The variables r and l represent the vertical and horizontal axes, respectively. The hyperbolic leaf function $sleafh_n(l)$ is the odd function. Therefore, it is obtained as follows:

$$sleafh_n(-l) = -sleafh_n(l) \quad (n = 1, 2, 3, \dots) \quad (36)$$

Without depending on the value of the basis n , the gradient around $l=0.0$ becomes 1.0 by the initial condition of Eq. (10).

Within the domain over $l=1.0$, the gradient dr/dl sharply increases according to increase in the basis n . Except for the basis $n=1$, the limit of the variable l exists in the hyperbolic leaf function $sleafh_n(l)$. The limit with respect to the basis n is defined as ζ_n (>0). The following equation is discussed:

$$\lim_{l \rightarrow \zeta_n} sleafh_n(l) = \infty \quad (\zeta_n > 0) \quad (n = 2, 3, 4, \dots) \quad (37)$$

The limit ζ_n with respect to the basis n is obtained by the following equation:

$$\zeta_n = \int_0^{\infty} \frac{1}{\sqrt{1+t^{2n}}} dt (=l) \quad (n = 2, 3, 4, \dots) \quad (38)$$

The limit values are summarized in Table 1. The constant:

$$l = \frac{\pi_2}{\sqrt{2}} \quad (39)$$

is substituted in Eq. (34) (see Appendix B) as follows:

$$\begin{aligned} \left(sleafh_2\left(\frac{\pi_2}{\sqrt{2}}\right) \right)^2 &= \frac{1 + \sqrt{1 - \left(sleaf_2\left(\sqrt{2} \cdot \frac{\pi_2}{\sqrt{2}}\right) \right)^4}}{\left(sleaf_2\left(\sqrt{2} \cdot \frac{\pi_2}{\sqrt{2}}\right) \right)^2} \\ &= \frac{1 + \sqrt{1 - (sleaf_2(\pi_2))^4}}{(sleaf_2(\pi_2))^2} = \frac{1 + \sqrt{1 - (0)^4}}{(0)^2} = \infty \end{aligned} \quad (40)$$

Based on the result of Eq. (40), the relation between the constant ζ_2 and the constant π_2 is obtained as follows:

$$l = \zeta_2 = \frac{\pi_2}{\sqrt{2}} \quad \left(= \int_0^{\infty} \frac{1}{\sqrt{1+t^4}} dt = \frac{2}{\sqrt{2}} \int_0^1 \frac{1}{\sqrt{1-t^4}} dt \right) \quad (41)$$

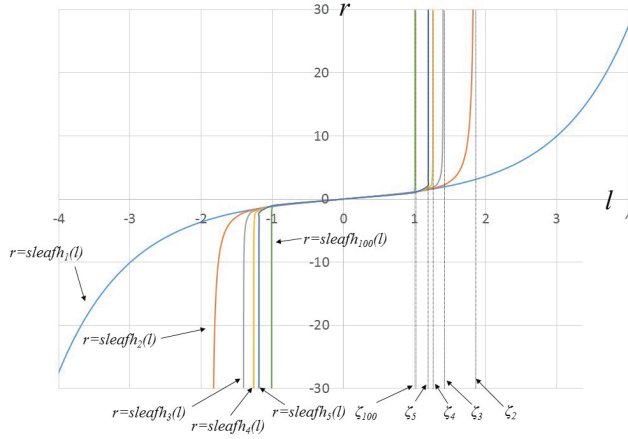


Fig. 1 Curves of the hyperbolic leaf function $sleafh_n(l)$.

Table 1. Limit ζ_n of variable l with respect to the hyperbolic leaf function $sleafh_n(l)$. (All results have been rounded to no more than six significant figures)

Limit ζ_n	Value
ζ_1	N/A
ζ_2	1.85407
ζ_3	1.40218
ζ_4	1.25946
ζ_5	1.19057
ζ_{100}	1.00703

7. Extended definition of Hyperbolic Leaf Function $sleafh_n(l)$

With respect to an arbitrary variable l , the value of the leaf function $sleafh_n(l)$ can be obtained. On the other hand, except for the basis: $n=1$, the hyperbolic leaf function $sleafh_n(l)$ only can be obtained within the domain of the variable:

$$-\zeta_n < l < \zeta_n \quad (n = 2, 3, 4, \dots) \quad (42)$$

Eqs. (33)–(35) are not necessarily satisfied with respect to the arbitrary variable l because the hyperbolic leaf function $sleafh_n(l)$ is not defined with respect to all domains of the variable l . Therefore, to satisfy the Eqs. (33)–(35) with respect to all domains of the variable l , the hyperbolic leaf function $sleafh_n(l)$ is defined as a multivalued function. The multiple outputs of the variable l are obtained by one input of the variable r . Eq. (14) is redefined as follows:

$$l = 2m \cdot \zeta_n + \int_0^r \frac{1}{\sqrt{1+t^{2n}}} dt \quad (43)$$

$$(n = 2, 3, 4, \dots \quad m = \dots, -3, -2, -1, 0, 1, 2, 3, \dots)$$

The parameter m represents the integer. Using Eq. (43), the graph of the hyperbolic leaf function is shown in Fig. 2–Fig. 6. The vertical and the horizontal axes represent the variables r and l in Eq. (16), respectively.

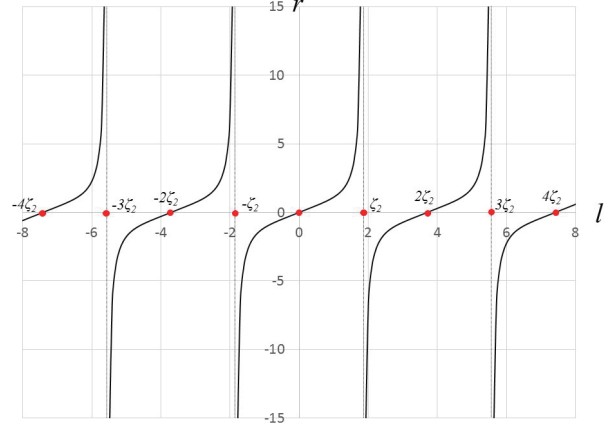


Fig. 2 Curve of the hyperbolic leaf function $r=sleafh_2(l)$.

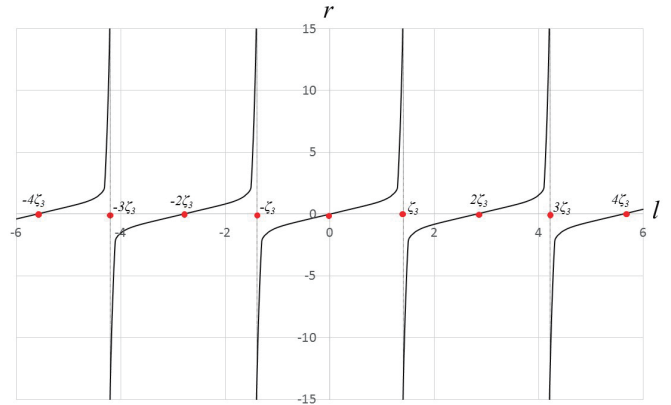


Fig. 3 Curve of the hyperbolic leaf function $r=sleafh_3(l)$.

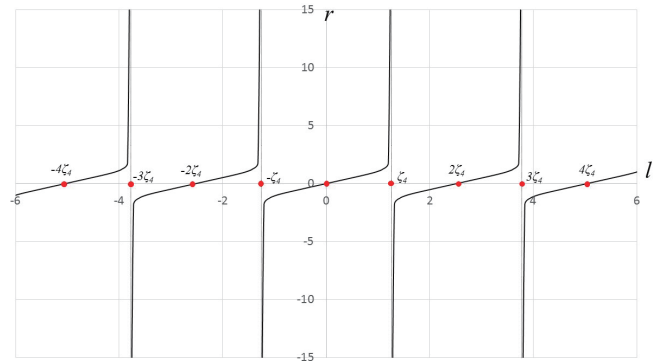


Fig. 4 Curve of the hyperbolic leaf function $r=sleafh_4(l)$.

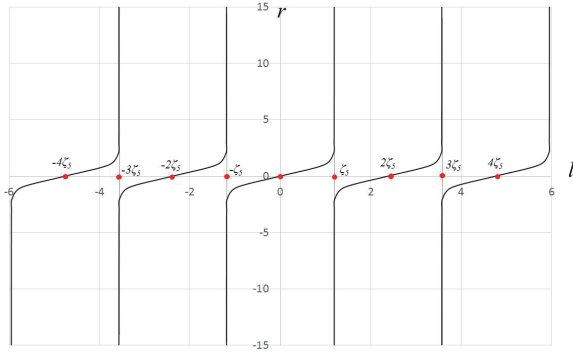


Fig. 5 Curve of the hyperbolic leaf function $r=sleafh_5(l)$.

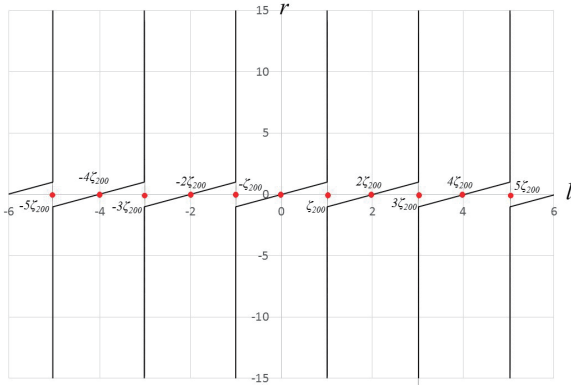


Fig. 6 Curve of the hyperbolic leaf function $r=sleafh_{100}(l)$.

8. Additional theorem of Hyperbolic Leaf Function $sleafh_2(l)$

The additional theorem of the hyperbolic leaf function $sleafh_2(l)$ is obtained as follows:

$$sleafh_2(l_1+l_2) = \frac{sleafh_2(l_1)\sqrt{1+(sleafh_2(l_2))^4} + sleafh_2(l_2)\sqrt{1+(sleafh_2(l_1))^4}}{1-(sleafh_2(l_1))^2(sleafh_2(l_2))^2} \quad (44)$$

$$sleafh_2(l_1-l_2) = \frac{sleafh_2(l_1)\sqrt{1+(sleafh_2(l_2))^4} - sleafh_2(l_2)\sqrt{1+(sleafh_2(l_1))^4}}{1-(sleafh_2(l_1))^2(sleafh_2(l_2))^2} \quad (45)$$

For detailed information, see Appendix D.

9. Conclusion

In the previous study, we discussed the ordinary differential equation, where the second derivative of a function is equal

to the negative value of the function with the power $2n-1$ (n : natural number). Periodicity was found in the ODE solutions of the leaf function.

In this study, we discuss the coupled ODE with respect to the leaf function, where the second derivative of a function is equal to the positive value of the function with the power $2n-1$ (n : natural number). Compared with the ODE in the previous study, the sign of the ODE in the hyperbolic leaf function in the present study is different. We conclude as follows:

- Using Maclaurin Series, the hyperbolic leaf function $sleafh_n(l)$ can be described by an infinite sum of polynomial terms.
- The hyperbolic leaf function $sleafh_n(l)$ has limits except for the function $sleafh_1(l)$.
- The relation between the leaf function $sleafh_n(l)$ and the hyperbolic leaf function $sleafh_n(l)$ is analogous to the relation between the trigonometric function $\sin(l)$ and the hyperbolic function $\sinh(l)$.

References

- [1] Kazunori Shinohara, Special function: Leaf function $r=sleafh_n(l)$ (First report), Bulletin of Daido University, **51**(2015), pp. 23–38.
- [2] Kazunori Shinohara, Special function: Leaf function $r=sleafh_n(l)$ (Second report), Bulletin of Daido University, **51**(2015), pp. 39–68.

Appendix A

In the case of $n=2, 3, 4, 5$, the derivative and the Maclaurin Series of the hyperbolic leaf function are described in this section. First, the hyperbolic leaf function $sleafh_2(l)$ is expanded as the Maclaurin Series. The first derivative of the hyperbolic leaf function $sleafh_2(l)$ is as follows:

$$\frac{d}{dl} sleafh_2(l) = \sqrt{1+(sleafh_2(l))^4} \quad (A1)$$

The second derivative of the hyperbolic leaf function $sleafh_2(l)$ is as follows:

$$\frac{d^2}{dl^2} sleafh_2(l) = 2 \cdot (sleafh_2(l))^3 \quad (A2)$$

The third derivative of the hyperbolic leaf function $sleafh_2(l)$ is as follows:

$$\frac{d^3}{dl^3} sleafh_2(l) = 6 \cdot (sleafh_2(l))^2 \cdot \sqrt{1 + (sleafh_2(l))^4} \quad (A3)$$

The fourth derivative of the hyperbolic leaf function $sleafh_2(l)$ is as follows:

$$\frac{d^4}{dl^4} sleafh_2(l) = 12 \cdot sleafh_2(l) \cdot (1 + 2(sleafh_2(l))^4) \quad (A4)$$

The fifth derivative of the hyperbolic leaf function $sleafh_2(l)$ is as follows:

$$\frac{d^5}{dl^5} sleafh_2(l) = 12 \cdot (1 + 10 \cdot (sleafh_2(l))^4) \sqrt{1 + (sleafh_2(l))^4} \quad (A5)$$

The sixth derivative of the hyperbolic leaf function $sleafh_2(l)$ is as follows:

$$\frac{d^6}{dl^6} sleafh_2(l) = 72 (sleafh_2(l))^3 (7 + 10(sleafh_2(l))^4) \quad (A6)$$

The seventh derivative of the hyperbolic leaf function $sleafh_2(l)$ is as follows:

$$\frac{d^7}{dl^7} sleafh_2(l) = 504 (sleafh_2(l))^2 (3 + 10(sleafh_2(l))^4) \sqrt{1 + (sleafh_2(l))^4} \quad (A7)$$

The eighth derivative of the hyperbolic leaf function $sleafh_2(l)$ is as follows:

$$\frac{d^8}{dl^8} sleafh_2(l) = 1008 sleafh_2(l) (3 + 36(sleafh_2(l))^4 + 40(sleafh_2(l))^8) \quad (A8)$$

The ninth derivative of the hyperbolic leaf function $sleafh_2(l)$ is as follows:

$$\begin{aligned} \frac{d^9}{dl^9} sleafh_2(l) &= 3024 (1 + 60(sleafh_2(l))^4 + 2(sleafh_2(l))^8) \sqrt{1 + (sleafh_2(l))^4} \end{aligned} \quad (A9)$$

The tenth derivative of the hyperbolic leaf function $sleafh_2(l)$ is as follows:

$$\frac{d^{10}}{dl^{10}} sleafh_2(l) = 6048 (sleafh_2(l))^3 \cdot (121 + 660(sleafh_2(l))^4 + 600(sleafh_2(l))^8) \quad (A10)$$

The eleventh derivative of the hyperbolic leaf function $sleafh_2(l)$ is as follows:

$$\frac{d^{11}}{dl^{11}} sleafh_2(l) = 199584 (sleafh_2(l))^2 \cdot (11 + 140(sleafh_2(l))^4 + 200(sleafh_2(l))^8) \sqrt{1 + (sleafh_2(l))^4} \quad (A11)$$

The twelfth derivative of the hyperbolic leaf function $sleafh_2(l)$ is as follows:

$$\frac{d^{12}}{dl^{12}} sleafh_2(l) = 399168 sleafh_2(l) \cdot (1 + 2(sleafh_2(l))^4 (221 + 780(sleafh_2(l))^4 + 600(sleafh_2(l))^8)) \quad (A12)$$

The thirteenth derivative of the hyperbolic leaf function $sleafh_2(l)$ is as follows:

$$\frac{d^{13}}{dl^{13}} sleafh_2(l) = 399168 \sqrt{1 + (sleafh_2(l))^4} \cdot (1 + 130(sleafh_2(l))^4 (17 + 108(sleafh_2(l))^4 + 120(sleafh_2(l))^8)) \quad (A13)$$

Using the derivatives of Eqs. (A1)–(A13), the Maclaurin Series of the hyperbolic leaf function $sleafh_2(l)$ is obtained as follows:

$$\begin{aligned} sleafh_2(l) &= sleafh_2(0) + \frac{1}{1!} \left(\frac{d}{dl} sleafh_2(0) \right) l + \frac{1}{2!} \left(\frac{d^2}{dl^2} sleafh_2(0) \right) l^2 \\ &+ \frac{1}{3!} \left(\frac{d^3}{dl^3} sleafh_2(0) \right) l^3 + \dots + \frac{1}{13!} \left(\frac{d^{13}}{dl^{13}} sleafh_2(0) \right) l^{13} + \dots \\ &= \frac{1}{1!} l + \frac{12}{5!} l^5 + \frac{3024}{9!} l^9 + \frac{4390848}{13!} l^{13} + O(l^{17}) \\ &= l + \frac{1}{10} l^5 + \frac{1}{120} l^9 + \frac{11}{15600} l^{13} + O(l^{17}) \end{aligned} \quad (A14)$$

The symbol O represents the Landau symbol. Using the above equation, the second derivative with respect to the variable l is obtained as follows:

$$\frac{d^2}{dl^2} sleafh_2(l) = 2l^5 + \frac{3}{5}l^7 + \frac{11}{100}l^{11} + O(l^{15}) \quad (A15)$$

Using Eq. (A14), the following equation is obtained:

$$\begin{aligned} 2 \cdot (sleafh_2(l))^3 &= 2 \cdot \left(l + \frac{1}{10}l^5 + \frac{1}{120}l^9 + \frac{11}{15600}l^{13} + O(l^{17}) \right)^3 \\ &= 2l^5 + \frac{3}{5}l^7 + \frac{11}{100}l^{11} + O(l^{15}) \end{aligned} \quad (A16)$$

Eq. (A15) is equal to Eq. (A16). Therefore, the hyperbolic leaf function $sleafh_2(l)$ satisfies Eq. (6). Next, in the case of $n=3$, the Maclaurin Series is applied to the hyperbolic leaf function $sleafh_3(l)$. The first derivative of the hyperbolic leaf function $sleafh_3(l)$ is as follows:

$$\frac{d}{dl} sleafh_3(l) = \sqrt{1 + (sleafh_3(l))^6} \quad (A17)$$

The second derivative of the hyperbolic leaf function $sleafh_3(l)$ is as follows:

$$\frac{d^2}{dl^2} sleafh_3(l) = 3 \cdot (sleafh_3(l))^5 \quad (A18)$$

The third derivative of the hyperbolic leaf function $sleafh_3(l)$ is as follows:

$$\frac{d^3}{dl^3} sleafh_3(l) = 15 \cdot (sleafh_3(l))^4 \cdot \sqrt{1 + (sleafh_3(l))^6} \quad (A19)$$

The fourth derivative of the hyperbolic leaf function $sleafh_3(l)$ is as follows:

$$\frac{d^4}{dl^4} sleafh_3(l) = 15 \cdot (sleafh_3(l))^3 \cdot \left(4 + 7(sleafh_3(l))^6 \right) \quad (A20)$$

The fifth derivative of the hyperbolic leaf function $sleafh_3(l)$ is as follows:

$$\begin{aligned} \frac{d^5}{dl^5} sleafh_3(l) \\ &= 45(sleafh_3(l))^2 \left(4 + 21(sleafh_3(l))^6 \right) \sqrt{1 + (sleafh_3(l))^6} \end{aligned} \quad (A21)$$

The sixth derivative of the hyperbolic leaf function $sleafh_3(l)$ is as follows:

$$\begin{aligned} \frac{d^6}{dl^6} sleafh_3(l) \\ &= 45sleafh_3(l) \left(8 + 188(sleafh_3(l))^6 + 231(sleafh_3(l))^{12} \right) \end{aligned} \quad (A22)$$

The seventh derivative of the hyperbolic leaf function $sleafh_3(l)$ is as follows:

$$\begin{aligned} \frac{d^7}{dl^7} sleafh_3(l) &= 45\sqrt{1 + (sleafh_3(l))^6} \cdot \\ &\left\{ 8 + 7(sleafh_3(l))^6 \left(188 + 429(sleafh_3(l))^6 \right) \right\} \end{aligned} \quad (A23)$$

The eighth derivative of the hyperbolic leaf function $sleafh_3(l)$ is as follows:

$$\begin{aligned} \frac{d^8}{dl^8} sleafh_3(l) &= 2025(sleafh_3(l))^5 \cdot \\ &\left\{ 176 + 7(sleafh_3(l))^6 \left(152 + 143(sleafh_3(l))^6 \right) \right\} \end{aligned} \quad (A24)$$

The ninth derivative of the hyperbolic leaf function $sleafh_3(l)$ is as follows:

$$\begin{aligned} \frac{d^9}{dl^9} sleafh_3(l) &= 22275(sleafh_3(l))^4 \sqrt{1 + (sleafh_3(l))^6} \cdot \\ &\left\{ 80 + 7(sleafh_3(l))^6 \left(152 + 221(sleafh_3(l))^6 \right) \right\} \end{aligned} \quad (A25)$$

The tenth derivative of the hyperbolic leaf function $sleafh_3(l)$ is as follows:

$$\begin{aligned} \frac{d^{10}}{dl^{10}} sleafh_3(l) &= 22275(sleafh_3(l))^3 \cdot \\ &\left\{ 320 + 7(sleafh_3(l))^6 \left(1600 + 5512(sleafh_3(l))^6 + 4199(sleafh_3(l))^{12} \right) \right\} \end{aligned} \quad (A26)$$

The eleventh derivative of the hyperbolic leaf function $sleafh_3(l)$ is as follows:

$$\begin{aligned} \frac{d^{11}}{dl^{11}} sleafh_3(l) &= 66825(sleafh_3(l))^2 \sqrt{1 + (sleafh_3(l))^6} \cdot \\ &\left(320 + 7(sleafh_3(l))^6 \right) \left(4800 + 27560(sleafh_3(l))^6 + 29393(sleafh_3(l))^{12} \right) \end{aligned} \quad (A27)$$

The twelfth derivative of the hyperbolic leaf function $sleafh_3(l)$ is as follows:

$$\begin{aligned} \frac{d^{12}}{dl^{12}}sleafh_3(l) &= 42768000\,sleafh_3(l) + 1806948000\,0\,(sleafh_3(l))^7 \\ &+ 2051848260\,00\,(sleafh_3(l))^{13} + 4941481545\,00\,(sleafh_3(l))^9 \\ &+ 3162341432\,25\,(sleafh_3(l))^{25} \end{aligned} \quad (A28)$$

The thirteenth derivative of the hyperbolic leaf function $sleafh_3(l)$ is as follows:

$$\begin{aligned} \frac{d^{13}}{dl^{13}}sleafh_3(l) &= 334125\sqrt{1+sleafh_3^6(l)} \cdot \\ &(128 + 378560\,(sleafh_3(l))^6 + 7983248\,(sleafh_3(l))^{12} \quad (A29) \\ &+ 28099708\,(sleafh_3(l))^{18} + 23661365\,(sleafh_3(l))^{24}) \end{aligned}$$

Using the derivatives of Eqs. (A17)–(A29), the Maclaurin Series of the hyperbolic leaf function $sleafh_3(l)$ is obtained as follows:

$$\begin{aligned} sleafh_3(l) &= \frac{1}{1!}l + \frac{360}{7!}l^7 + \frac{42768000}{13!}l^{13} \\ &+ \frac{9108557568\,0000}{19!}l^{19} + O(l^{25}) \quad (A30) \\ &= l + \frac{1}{14}l^7 + \frac{5}{728}l^{13} + \frac{145}{193648}l^{19} + O(l^{25}) \end{aligned}$$

Using the above equation, the second derivative with respect to the variable l is obtained as follows:

$$\frac{d^2}{dl^2}sleafh_3(l) = 3l^5 + \frac{15}{14}l^{11} + \frac{1305}{5096}l^{17} + O(l^{23}) \quad (A31)$$

Using Eq. (A30), the following equation is obtained:

$$\begin{aligned} 3 \cdot (sleafh_3(l))^5 &= 3 \cdot \left(l + \frac{1}{14}l^7 + \frac{5}{728}l^{13} + \frac{145}{193648}l^{19} + O(l^{25}) \right)^5 \\ &= 3l^5 + \frac{15}{14}l^{11} + \frac{1305}{5096}l^{17} + O(l^{23}) \end{aligned} \quad (A32)$$

Eq. (A31) is equal to Eq. (A32). Therefore, the hyperbolic leaf function $sleafh_3(l)$ satisfies Eq. (6). Next, in the case of

the basis: $n=4$, the Maclaurin Series is applied to the hyperbolic leaf function $sleafh_4(l)$. The first derivative of the hyperbolic leaf function $sleafh_4(l)$ is as follows:

$$\frac{d}{dl}sleafh_4(l) = \sqrt{1 + (sleafh_4(l))^8} \quad (A33)$$

The second derivative of the hyperbolic leaf function $sleafh_4(l)$ is as follows:

$$\frac{d^2}{dl^2}sleafh_4(l) = 4 \cdot (sleafh_4(l))^7 \quad (A34)$$

The third derivative of the hyperbolic leaf function $sleafh_4(l)$ is as follows:

$$\frac{d^3}{dl^3}sleafh_4(l) = 28 \cdot (sleafh_4(l))^6 \cdot \sqrt{1 + (sleafh_4(l))^8} \quad (A35)$$

The fourth derivative of the hyperbolic leaf function $sleafh_4(l)$ is as follows:

$$\frac{d^4}{dl^4}sleafh_4(l) = 56 \cdot (sleafh_4(l))^5 \cdot \left(3 + 5(sleafh_4(l))^8 \right) \quad (A36)$$

The fifth derivative of the hyperbolic leaf function $sleafh_4(l)$ is as follows:

$$\begin{aligned} \frac{d^5}{dl^5}sleafh_4(l) &= 280(sleafh_4(l))^4 \cdot \\ &\left(3 + 13(sleafh_4(l))^8 \right) \sqrt{1 + (sleafh_4(l))^8} \end{aligned} \quad (A37)$$

The sixth derivative of the hyperbolic leaf function $sleafh_4(l)$ is as follows:

$$\begin{aligned} \frac{d^6}{dl^6}sleafh_4(l) &= 1120(sleafh_4(l))^3 \cdot \\ &\left(3 + 45(sleafh_4(l))^8 + 52(sleafh_4(l))^6 \right) \end{aligned} \quad (A38)$$

The seventh derivative of the hyperbolic leaf function $sleafh_4(l)$ is as follows:

$$\begin{aligned} \frac{d^7}{dl^7}sleafh_4(l) &= 1120(sleafh_4(l))^2 \sqrt{1 + (sleafh_4(l))^8} \cdot \\ &\left(9 + 495(sleafh_4(l))^8 + 988(sleafh_4(l))^6 \right) \end{aligned} \quad (A39)$$

The eighth derivative of the hyperbolic leaf function

$sleafh_4(l)$ is as follows:

$$\begin{aligned} \frac{d^8}{dl^8} sleafh_4(l) &= 2240 sleafh_4(l) \cdot \\ &\left(9 + 2502(sleafh_4(l))^8 + 12357(sleafh_4(l))^6 + 10868(sleafh_4(l))^{24}\right) \end{aligned} \quad (\text{A40})$$

The ninth derivative of the hyperbolic leaf function $sleafh_4(l)$ is as follows:

$$\begin{aligned} \frac{d^9}{dl^9} sleafh_4(l) &= 2240 \sqrt{1 + (sleafh_4(l))^8} \cdot \\ &\left(9 + 22518(sleafh_4(l))^8 + 210069(sleafh_4(l))^6 + 271700(sleafh_4(l))^{24}\right) \end{aligned} \quad (\text{A41})$$

The tenth derivative of the hyperbolic leaf function $sleafh_4(l)$ is as follows:

$$\begin{aligned} \frac{d^{10}}{dl^{10}} sleafh_4(l) &= 313600 (sleafh_4(l))^7 \cdot \\ &\left(1287 + 25938(sleafh_4(l))^8 + 76587(sleafh_4(l))^6 + 54340(sleafh_4(l))^{24}\right) \end{aligned} \quad (\text{A42})$$

The eleventh derivative of the hyperbolic leaf function $sleafh_4(l)$ is as follows:

$$\begin{aligned} \frac{d^{11}}{dl^{11}} sleafh_4(l) &= 313600 (sleafh_4(l))^6 \sqrt{1 + (sleafh_4(l))^8} \cdot \\ &\left(9009 + 389070(sleafh_4(l))^8 + 1761501(sleafh_4(l))^6 + 1684540(sleafh_4(l))^{24}\right) \end{aligned} \quad (\text{A43})$$

The twelfth derivative of the hyperbolic leaf function $sleafh_4(l)$ is as follows:

$$\begin{aligned} \frac{d^{12}}{dl^{12}} sleafh_4(l) &= 627200 (sleafh_4(l))^5 \cdot \\ &\left(27027 + 17(sleafh_4(l))^8 + 13(sleafh_4(l))^6 + 103521 + 217953(sleafh_4(l))^8 + 129580(sleafh_4(l))^{24}\right) \end{aligned} \quad (\text{A44})$$

The thirteenth derivative of the hyperbolic leaf function $sleafh_4(l)$ is as follows:

$$\begin{aligned} \frac{d^{13}}{dl^{13}} sleafh_4(l) &= 8153600 (sleafh_4(l))^4 \sqrt{1 + (sleafh_4(l))^8} \cdot \left\{0395 + 17(sleafh_4(l))^8 \cdot \right. \\ &\left. (162855 + 2173941(sleafh_4(l))^8 + 6320637(sleafh_4(l))^6 + 4794460(sleafh_4(l))^{24})\right\} \end{aligned} \quad (\text{A45})$$

Using the derivatives of Eqs. (A33)–(A45), the Maclaurin Series of the hyperbolic leaf function $sleafh_4(l)$ is obtained

as follows:

$$sleafh_4(l) = l + \frac{1}{18}l^9 + \frac{7}{1224}l^{17} + \frac{77}{110160}l^{25} + O(l^{33}) \quad (\text{A46})$$

Using the above equation, the second derivative with respect to the variable l is obtained as follows:

$$\frac{d^2}{dl^2} sleafh_4(l) = 4l^7 + \frac{14}{9}l^{15} + \frac{385}{918}l^{23} + O(l^{31}) \quad (\text{A47})$$

Using Eq. (A46), the following equation is obtained:

$$\begin{aligned} 4 \cdot (sleafh_4(l))^7 &= 4 \cdot \left(l + \frac{1}{18}l^9 + \frac{7}{1224}l^{17} + \frac{77}{110160}l^{25} + O(l^{33}) \right)^7 \\ &= 4l^7 + \frac{14}{9}l^{15} + \frac{385}{918}l^{23} + O(l^{31}) \end{aligned} \quad (\text{A48})$$

Eq. (A47) is equal to Eq. (A48). Therefore, the hyperbolic leaf function $sleafh_4(l)$ satisfies Eq. (6). Next, in the case of the basis: $n=5$, the Maclaurin Series is applied to the hyperbolic leaf function $sleafh_5(l)$. The first derivative of the hyperbolic leaf function $sleafh_5(l)$ is as follows:

$$\frac{d}{dl} sleafh_5(l) = \sqrt{1 + (sleafh_5(l))^{10}} \quad (\text{A49})$$

The second derivative of the hyperbolic leaf function $sleafh_5(l)$ is as follows:

$$\frac{d^2}{dl^2} sleafh_5(l) = 5 \cdot (sleafh_5(l))^9 \quad (\text{A50})$$

The third derivative of the hyperbolic leaf function $sleafh_5(l)$ is as follows:

$$\frac{d^3}{dl^3} sleafh_5(l) = 45 \cdot (sleafh_5(l))^8 \cdot \sqrt{1 + (sleafh_5(l))^{10}} \quad (\text{A51})$$

The fourth derivative of the hyperbolic leaf function $sleafh_5(l)$ is as follows:

$$\frac{d^4}{dl^4} sleafh_5(l) = 45 \cdot (sleafh_5(l))^7 \cdot \left(8 + 13(sleafh_5(l))^{10}\right) \quad (\text{A52})$$

The fifth derivative of the hyperbolic leaf function $sleafh_5(l)$ is as follows:

$$\frac{d^5}{dl^5} sleafh_5(l) = 45(sleafh_5(l))^6 \cdot \sqrt{56 + 221(sleafh_5(l))^{10} + (sleafh_5(l))^{20}} \quad (A53)$$

The sixth derivative of the hyperbolic leaf function $sleafh_5(l)$ is as follows:

$$\frac{d^6}{dl^6} sleafh_5(l) = 135(sleafh_5(l))^5 \cdot (112 + 1384(sleafh_5(l))^{10} + 1547(sleafh_5(l))^{20}) \quad (A54)$$

The seventh derivative of the hyperbolic leaf function $sleafh_5(l)$ is as follows:

$$\frac{d^7}{dl^7} sleafh_5(l) = 675(sleafh_5(l))^4 \sqrt{112 + 4152(sleafh_5(l))^{10} + 7735(sleafh_5(l))^{20}} \cdot \sqrt{1 + (sleafh_5(l))^{10}} \quad (A55)$$

The eighth derivative of the hyperbolic leaf function $sleafh_5(l)$ is as follows:

$$\frac{d^8}{dl^8} sleafh_5(l) = 675(sleafh_5(l))^3 \cdot (448 + 59136(sleafh_5(l))^{10} + 264528(sleafh_5(l))^{20} + 224315(sleafh_5(l))^{30}) \quad (A56)$$

The ninth derivative of the hyperbolic leaf function $sleafh_5(l)$ is as follows:

$$\frac{d^9}{dl^9} sleafh_5(l) = 2025(sleafh_5(l))^2 \sqrt{448 + 256256(sleafh_5(l))^{10} + 202804(sleafh_5(l))^{20} + 2467465(sleafh_5(l))^{30}} \cdot \sqrt{1 + (sleafh_5(l))^{10}} \quad (A57)$$

The tenth derivative of the hyperbolic leaf function $sleafh_5(l)$ is as follows:

$$\frac{d^{10}}{dl^{10}} sleafh_5(l) = 2025 sleafh_5(l) \cdot (896 + 3078208(sleafh_5(l))^{10} + 48973408(sleafh_5(l))^{20} + 133716176(sleafh_5(l))^{30} + 91296205(sleafh_5(l))^{40}) \quad (A58)$$

The eleventh derivative of the hyperbolic leaf function $sleafh_5(l)$ is as follows:

$$\frac{d^{11}}{dl^{11}} sleafh_5(l) = 2025 \sqrt{896 + 11(sleafh_5(l))^{10} + 3078208 + 93494688(sleafh_5(l))^{10} + 376836496(sleafh_5(l))^{20} + 340285855(sleafh_5(l))^{30}} \cdot \sqrt{1 + (sleafh_5(l))^{10}} \quad (A59)$$

Using the derivatives of Eqs. (A49)–(A59), the Maclaurin Series of the hyperbolic leaf function $sleafh_5(l)$ is obtained as follows:

$$sleafh_5(l) = l + \frac{1}{22} l^{11} + \frac{3}{616} l^{21} + \frac{267}{420112} l^{31} + \frac{136545}{1515764096} l^{41} + O(l^{51}) \quad (A60)$$

Using the above equation, the second derivative with respect to the variable l is obtained as follows:

$$\frac{d^2}{dl^2} sleafh_5(l) = 5l^9 + \frac{45}{22} l^{19} + \frac{4005}{6776} l^{29} + \frac{682725}{4621232} l^{39} + O(l^{49}) \quad (A61)$$

Using Eq. (A60), the following equation is obtained:

$$\begin{aligned} & 5 \cdot (sleafh_5(l))^9 \\ &= 5 \cdot \left(l + \frac{1}{22} l^{11} + \frac{3}{616} l^{21} + \frac{267}{420112} l^{31} + \frac{136545}{1515764096} l^{41} + O(l^{51}) \right)^9 \\ &= 5l^9 + \frac{45}{22} l^{19} + \frac{4005}{6776} l^{29} + \frac{682725}{4621232} l^{39} + O(l^{49}) \end{aligned} \quad (A62)$$

Eq. (A61) is equal to Eq. (A62). Therefore, the hyperbolic leaf function $sleafh_5(l)$ satisfies Eq. (6).

Appendix B

In this section, the relation between the leaf function $sleafh_2(l)$ and the hyperbolic leaf function $sleafh_2(l)$ is described. The following polynomial is considered.

$$y^2 x^4 - 2x^2 + y^2 = 0 \quad (B1)$$

The above equation is transformed as the following:

$$y = \pm \frac{\sqrt{2x}}{\sqrt{1+x^4}} \quad (B2)$$

We only discuss about the following equation.

$$y = - \frac{\sqrt{2x}}{\sqrt{1+x^4}} \quad (B3)$$

The above equation is derived with respect to the variable x .

$$\frac{dy}{dx} = -\sqrt{2} \frac{(1-x^4)}{(1+x^4)^{\frac{3}{2}}} \quad (\text{B4})$$

On the other hand, the following equation is obtained.

$$\begin{aligned} \frac{1}{\sqrt{1-y^4}} \frac{dy}{dx} &= \frac{1}{\sqrt{1-\left(-\frac{\sqrt{2}x}{\sqrt{1+x^4}}\right)^4}} \left\{ -\sqrt{2} \frac{(1-x^4)}{(1+x^4)^{\frac{3}{2}}} \right\} \\ &= -\frac{1+x^4}{x^4-1} \sqrt{2} \frac{(1-x^4)}{(1+x^4)^{\frac{3}{2}}} = \sqrt{2} \frac{1}{\sqrt{x^4+1}} \end{aligned} \quad (\text{B5})$$

Using the above equation, the following equation is obtained.

$$\frac{dy}{\sqrt{1-y^4}} - \sqrt{2} \frac{dx}{\sqrt{x^4+1}} = 0 \quad (\text{B6})$$

where the variables x and y are set as the following equations:

$$x = \text{sleaf}h_2(l) \quad (\text{B7})$$

$$y = \text{sleaf}_2(\sqrt{2}l) \quad (\text{B8})$$

The above equation is differentiated with respect to the variable l .

$$\frac{dx}{dl} = \sqrt{1+(\text{sleaf}h_2(l))^4} = \sqrt{1+x^4} \quad (\text{B9})$$

$$\frac{dy}{dl} = \sqrt{2} \sqrt{1-(\text{sleaf}_2(\sqrt{2}l))^4} = \sqrt{2} \sqrt{1-y^4} \quad (\text{B10})$$

Using the above equations, the following equation is obtained:

$$\begin{aligned} \frac{dy}{\sqrt{1-y^4}} - \sqrt{2} \frac{dx}{\sqrt{x^4+1}} &= \frac{1}{\sqrt{1-y^4}} \frac{dy}{dl} - \sqrt{2} \frac{1}{\sqrt{x^4+1}} \frac{dx}{dl} dl \\ &= \frac{1}{\sqrt{1-y^4}} \sqrt{2} \sqrt{1-y^4} dl - \sqrt{2} \frac{1}{\sqrt{x^4+1}} \sqrt{x^4+1} dl = 0 \end{aligned} \quad (\text{B11})$$

Eq. (B7) and Eq. (B8) satisfy Eq. (B6). Therefore, the following relation is obtained by Eq. (B1).

$$\left(\text{sleaf}_2(\sqrt{2}l)\right)^2 (\text{sleaf}h_2(l))^4 - 2(\text{sleaf}h_2(l))^2 + \left(\text{sleaf}_2(\sqrt{2}l)\right)^2 = 0 \quad (\text{B12})$$

Solving the above equation for the hyperbolic leaf function $\text{sleaf}h_2(l)$, the two solutions can be obtained. In the case of inequality $|\text{sleaf}h_2(l)| \leq 1$, the following equation is applied.

$$\left(\text{sleaf}h_2(l)\right)^2 = \frac{1 - \sqrt{1 - \left(\text{sleaf}_2(\sqrt{2} \cdot l)\right)^4}}{\left(\text{sleaf}_2(\sqrt{2} \cdot l)\right)^2} \quad (\text{B13})$$

In the case of the inequality $|\text{sleaf}h_2(l)| > 1$, the following equation is applied.

$$\left(\text{sleaf}h_2(l)\right)^2 = \frac{1 + \sqrt{1 - \left(\text{sleaf}_2(\sqrt{2} \cdot l)\right)^4}}{\left(\text{sleaf}_2(\sqrt{2} \cdot l)\right)^2} \quad (\text{B14})$$

Appendix C

In this section, the relation between the hyperbolic function $\sinh(l)$ ($=\text{sleaf}h_1(l)$) and the hyperbolic leaf function $\text{sleaf}h_n(l)$ is described. The following equation is considered.

$$\left(\text{sleaf}h_n(l)\right)^n = \sinh(n\theta) \quad n=1, 2, 3, \dots \quad (\text{C1})$$

Using the above equation, the following equation is obtained.

$$\begin{aligned}\theta &= \frac{1}{n} \arcsin h\left(\left(\text{sleafh}_n(l)\right)^n\right) \\ &= \frac{1}{n} \ln\left(\left(\text{sleafh}_n(l)\right)^n + \sqrt{1 + \left(\text{sleafh}_n(l)\right)^{2n}}\right) \\ n &= 1, 2, 3, \dots\end{aligned}\quad (C2)$$

The above equation is differentiated with respect to the variable l .

$$\begin{aligned}\frac{d\theta}{dl} &= \frac{n\left(\text{sleafh}_n(l)\right)^{n-1}}{n\sqrt{1 + \left(\text{sleafh}_n(l)\right)^{2n}}} \sqrt{1 + \left(\text{sleafh}_n(l)\right)^{2n}} \\ &= \left(\text{sleafh}_n(l)\right)^{n-1}\end{aligned}\quad (C3)$$

The following equation is obtained by integrating the above equation from 0 to l .

$$\theta = \int_0^l \left(\text{sleafh}_n(t)\right)^{n-1} dt \quad (l=0, \theta=0 \text{ by (C2)}) \quad (C4)$$

Using Eqs. (C1) and (C4), the following equation is obtained.

$$\begin{aligned}\left(\text{sleafh}_n(l)\right)^n &= \sinh\left(n \int_0^l \left(\text{sleafh}_n(t)\right)^{n-1} dt\right) \\ n &= 1, 2, 3, \dots\end{aligned}\quad (C5)$$

Appendix D

To prove the additional theorem of Eq. (44), the following equation is set.

$$l_1 + l_2 = c \quad (D1)$$

The symbol c represents the arbitrary constant. Using Eqs. (D1) and (44), the following equation is obtained.

$$\begin{aligned}\text{sleafh}_2(c) &= \\ \frac{\text{sleafh}_2(l_1)\sqrt{1 + \left(\text{sleafh}_2(c-l_1)\right)^4} + \text{sleafh}_2(c-l_1)\sqrt{1 + \left(\text{sleafh}_2(l_1)\right)^4}}{1 - \left(\text{sleafh}_2(l_1)\right)^2 \left(\text{sleafh}_2(c-l_1)\right)^2}\end{aligned}\quad (D2)$$

The right side of the above equation is defined as follows:

$$F(l_1) = \frac{\text{sleafh}_2(l_1)\sqrt{1 + \left(\text{sleafh}_2(c-l_1)\right)^4} + \text{sleafh}_2(c-l_1)\sqrt{1 + \left(\text{sleafh}_2(l_1)\right)^4}}{1 - \left(\text{sleafh}_2(l_1)\right)^2 \left(\text{sleafh}_2(c-l_1)\right)^2} \quad (D3)$$

The symbol $\text{sleafh}_2(c)$ is just constant. The following equation is obtained by Eqs. (D2) and (D3).

$$F(l_1) = \text{sleafh}_2(c) \quad (D4)$$

Therefore, the function $F(l_1)$ also has to be a constant.

$$\frac{\partial F(l_1)}{\partial l_1} = 0 \quad (D5)$$

If the above equation is satisfied, the function $F(l_1)$ becomes a constant. To prove Eq. (D4), the function $F(l_1)$ is differentiated with respect to the variable l_1 .

$$\begin{aligned}\frac{\partial F(l_1)}{\partial l_1} &= \\ \frac{\left\{\text{sleafh}_2(l_1)\sqrt{1 + \left(\text{sleafh}_2(c-l_1)\right)^4} + \text{sleafh}_2(c-l_1)\sqrt{1 + \left(\text{sleafh}_2(l_1)\right)^4}\right\} \left\{1 - \left(\text{sleafh}_2(l_1)\right)^2 \left(\text{sleafh}_2(c-l_1)\right)^2\right\} - \left\{\text{sleafh}_2(l_1)\sqrt{1 + \left(\text{sleafh}_2(c-l_1)\right)^4} + \text{sleafh}_2(c-l_1)\sqrt{1 + \left(\text{sleafh}_2(l_1)\right)^4}\right\} \left\{1 - \left(\text{sleafh}_2(l_1)\right)^2 \left(\text{sleafh}_2(c-l_1)\right)^2\right\}'}{\left\{1 - \left(\text{sleafh}_2(l_1)\right)^2 \left(\text{sleafh}_2(c-l_1)\right)^2\right\}^2}\end{aligned}\quad (D6)$$

On the other hand, the following equation is obtained.

$$\begin{aligned}\left\{\text{sleafh}_2(l_1)\sqrt{1 + \left(\text{sleafh}_2(c-l_1)\right)^4} + \text{sleafh}_2(c-l_1)\sqrt{1 + \left(\text{sleafh}_2(l_1)\right)^4}\right\}' \\ = \sqrt{1 + \left(\text{sleafh}_2(l_1)\right)^4} \sqrt{1 + \left(\text{sleafh}_2(c-l_1)\right)^4} - 2\text{sleafh}_2(l_1)\text{sleafh}_2(c-l_1) \\ - \sqrt{1 + \left(\text{sleafh}_2(c-l_1)\right)^4} \sqrt{1 + \left(\text{sleafh}_2(l_1)\right)^4} + 2\text{sleafh}_2(c-l_1)\text{sleafh}_2(l_1) \\ = 2\text{sleafh}_2(c-l_1)\text{sleafh}_2(l_1) - 2\text{sleafh}_2(l_1)\text{sleafh}_2(c-l_1)\end{aligned}\quad (D7)$$

$$\begin{aligned}\left\{1 - \left(\text{sleafh}_2(l_1)\right)^2 \left(\text{sleafh}_2(c-l_1)\right)^2\right\}' \\ = -2\left(\text{sleafh}_2(l_1)\right)\left(\text{sleafh}_2(c-l_1)\right)^2 \sqrt{1 + \left(\text{sleafh}_2(l_1)\right)^4} \\ + 2\left(\text{sleafh}_2(l_1)\right)^2 \text{sleafh}_2(c-l_1) \sqrt{1 + \left(\text{sleafh}_2(c-l_1)\right)^4}\end{aligned}\quad (D8)$$

By substituting Eqs. (D7) and (D8) into Eq. (D6), Eq. (D5) is obtained. The function $F(l_1)$ does not depend on the variable l_1 . Therefore, the following equation is obtained.

$$F(l_1) = F(0) \quad (D9)$$

By substituting 0 into Eq. (D3), the following equation is obtained.

$$\begin{aligned}F(0) &= \frac{\text{sleafh}_2(0)\sqrt{1 + \left(\text{sleafh}_2(c-0)\right)^4} + \text{sleafh}_2(c-0)\sqrt{1 + \left(\text{sleafh}_2(0)\right)^4}}{1 - \left(\text{sleafh}_2(0)\right)^2 \left(\text{sleafh}_2(c-0)\right)^2} \\ &= \text{sleafh}_2(c)\end{aligned}\quad (D10)$$

By Eqs. (D9) (D10), Eq. (D4) is obtained. The proof of Eq. (45) is similar with that of Eq. (44).

Appendix E

The integration of the hyperbolic leaf function: $(sleafh_n(l))^{n-1}$ is obtained as follows:

$$\begin{aligned} & n \int_0^l (sleafh_n(t))^{n-1} dt \\ &= \ln \left((sleafh_n(l))^n + \sqrt{1 + (sleafh_n(l))^{2n}} \right) \\ & 0 \leq l < \zeta_n \\ & n = 2, 3, \dots \end{aligned} \quad (E1)$$

The proof is as follows :

$$\begin{aligned} & \frac{d}{dl} \left(\sqrt{1 + (sleafh_n(l))^{2n}} \right) = \frac{d}{dl} \left(1 + (sleafh_n(l))^{2n} \right)^{\frac{1}{2}} \\ &= \frac{1}{2} \left(1 + (sleafh_n(l))^{2n} \right)^{-\frac{1}{2}} \cdot 2n (sleafh_n(l))^{2n-1} \cdot \sqrt{1 + (sleafh_n(l))^{2n}} \\ &= n \frac{(sleafh_n(l))^{2n-1} \sqrt{1 + (sleafh_n(l))^{2n}}}{\sqrt{1 + (sleafh_n(l))^{2n}}} \\ &= n (sleafh_n(l))^{2n-1} \end{aligned} \quad (E2)$$

Using Eq. (E2), the following equation is obtained:

$$\begin{aligned} & \frac{d}{dl} \ln \left((sleafh_n(l))^n + \sqrt{1 + (sleafh_n(l))^{2n}} \right) \\ &= \frac{n (sleafh_n(l))^{n-1} \sqrt{1 + (sleafh_n(l))^{2n}} + n (sleafh_n(l))^{2n-1}}{(sleafh_n(l))^n + \sqrt{1 + (sleafh_n(l))^{2n}}} \\ &= n (sleafh_n(l))^{n-1} \frac{\sqrt{1 + (sleafh_n(l))^{2n}} + (sleafh_n(l))^n}{(sleafh_n(l))^n + \sqrt{1 + (sleafh_n(l))^{2n}}} \\ &= n (sleafh_n(l))^{n-1} \end{aligned} \quad (E3)$$

In the case $n=1$ of Eq. (E1), the following equation is obtained:

$$\begin{aligned} & \int_0^l (sleafh_1(t))^0 dt \\ &= \ln \left((sleafh_1(l))^1 + \sqrt{1 + (sleafh_1(l))^2} \right) \end{aligned} \quad (E4)$$

$$\int_0^l dt = \ln \left(sleafh_1(l) + \sqrt{1 + (sleafh_1(l))^2} \right) \quad (E5)$$

$$l = \ln \left(sleafh_1(l) + \sqrt{1 + (sleafh_1(l))^2} \right) \quad (E6)$$

Therefore, the following equation is obtained:

$$e^l = sleafh_1(l) + \sqrt{1 + (sleafh_1(l))^2} \quad (E7)$$

Using Eq. (17), the above equation represents the following equation:

$$e^l = \cosh(l) + \sinh(l) \quad (E8)$$

Appendix F

The numerical data of the hyperbolic leaf function is summarized in the succeeding Tables.

Table 2 Numerical data of the hyperbolic leaf function $sleafh_n(l)$.
 (All results have been rounded to no more than five significant figures)

l	$r (=sleafh_n(l))$					
	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=100$
-5.4	-110.70	-6.1639	0.2087	-0.3621	-0.6380	0.6422
-5.2	-90.633	-2.7559	0.4088	-0.1621	-0.4376	0.8422
-5.0	-74.203	-1.7609	0.6109	0.0378	-0.2376	-0.9718
-4.8	-60.751	-1.2679	0.8253	0.2378	-0.0376	-0.7718
-4.6	-49.737	-0.9514	1.0933	0.4378	0.1623	-0.5718
-4.4	-40.719	-0.7080	1.6018	0.6388	0.3623	-0.3718
-4.2	-33.335	-0.4947	-8.7394	0.8494	0.5624	-0.1718
-4.0	-27.289	-0.2920	-1.5490	1.1283	0.7646	0.0281
-3.8	-22.339	-0.0918	-1.0721	2.4892	0.9945	0.2281
-3.6	-18.285	0.1081	-0.8104	-1.2211	1.7243	0.4281
-3.4	-14.965	0.3084	-0.5975	-0.8995	-1.0832	0.6281
-3.2	-12.245	0.5115	-0.3957	-0.6828	-0.8239	0.8281
-3.0	-10.017	0.7263	-0.1956	-0.4811	-0.6190	-0.9860
-2.8	-8.1919	0.9736	0.0043	-0.2810	-0.4188	-0.7859
-2.6	-6.6947	1.2993	0.2043	-0.0810	-0.2188	-0.5859
-2.4	-5.4662	1.8155	0.4045	0.1189	-0.0188	-0.3859
-2.2	-4.4571	2.8866	0.6064	0.3189	0.1811	-0.1859
-2.0	-3.6268	6.8525	0.8203	0.5190	0.3811	0.0140
-1.8	-2.9421	-18.492	1.0862	0.7218	0.5812	0.2140
-1.6	-2.3755	-3.9342	1.5837	0.9463	0.7842	0.4140
-1.4	-1.9043	-2.1929	-15.137	1.3275	1.0217	0.6140
-1.2	-1.5094	-1.5009	-1.5660	-1.7755	2.2696	0.8140
-1.0	-1.1752	-1.1091	-1.0791	-1.0620	-1.0510	-1.0028
-0.8	-0.8881	-0.8339	-0.8153	-0.8075	-0.8039	-0.8000
-0.6	-0.6366	-0.6078	-0.6020	-0.6005	-0.6001	-0.6000
-0.4	-0.4107	-0.4010	-0.4001	-0.4000	-0.4000	-0.4000
-0.2	-0.2013	-0.2000	-0.2000	-0.2000	-0.2000	-0.2000
0.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 3 Numerical data of the hyperbolic leaf function $sleafh_n(l)$.
 (All results have been rounded to no more than five significant figures)

l	$r (=sleafh_n(l))$					
	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=100$
0.2	0.2013	0.2000	0.2000	0.2000	0.2000	0.2000
0.4	0.4107	0.4010	0.4001	0.4000	0.4000	0.4000
0.6	0.6366	0.6078	0.6020	0.6005	0.6001	0.6000
0.8	0.8881	0.8339	0.8153	0.8075	0.8039	0.8000
1.0	1.1752	1.1091	1.0791	1.0620	1.0510	1.0028
1.2	1.5094	1.5009	1.5660	1.7756	-4.8024	-0.8140
1.4	1.9043	2.1929	15.137	-1.3275	-1.0216	-0.6140
1.6	2.3755	3.9342	-1.5836	-0.9463	-0.7841	-0.4140
1.8	2.9421	18.492	-1.0861	-0.7217	-0.5812	-0.2140
2.0	3.6268	-6.8525	-0.8203	-0.5190	-0.3811	-0.0140
2.2	4.4571	-2.8866	-0.6064	-0.3189	-0.1811	0.1859
2.4	5.4662	-1.8155	-0.4044	-0.1189	0.0188	0.3859
2.6	6.6947	-1.2993	-0.2043	0.0810	0.2188	0.5859
2.8	8.1919	-0.9736	-0.0043	0.2810	0.4188	0.7859
3.0	10.017	-0.7263	0.1956	0.4811	0.6190	0.9860
3.2	12.245	-0.5115	0.3957	0.6828	0.8239	-0.8281
3.4	14.965	-0.3084	0.5975	0.8995	1.0832	-0.6281
3.6	18.285	-0.1081	0.8104	1.2211	-1.7243	-0.4281
3.8	22.339	0.0918	1.0721	-2.4892	-0.9944	-0.2281
4.0	27.289	0.2920	1.5490	-1.1283	-0.7646	-0.0281
4.2	33.335	0.4947	8.7395	-0.8494	-0.5624	0.1718
4.4	40.719	0.7080	-1.6018	-0.6388	-0.3623	0.3718
4.6	49.737	0.9514	-1.0933	-0.4378	-0.1623	0.5718
4.8	60.751	1.2679	-0.8253	-0.2378	0.0376	0.7718
5.0	74.203	1.7609	-0.6109	-0.0378	0.2376	0.9718
5.2	90.633	2.7559	-0.4088	0.1621	0.4376	-0.8422
5.4	110.70	6.1639	-0.2087	0.3621	0.6380	-0.6422