Special Function: Hyperbolic Leaf Function $r = cleafh_n(l)$ (Second Report)

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Summary

In previous reports, the leaf function $sleafh_n(l)$ is defined. This function is satisfied by the ordinary differential equation (ODE) and the following initial conditions:

$$\frac{d^2r(l)}{dl^2} = n \cdot r(l)^{2n-1} \quad n = 1,2,3,\cdots$$
$$r(0) = 0$$
$$\frac{dr(0)}{dl} = 1$$

Variable $r(l)$ consisting of parameter l represents the hyperbolic leaf functions. Parameter n represents the basis (the natural number). In the case of the basis $n = 1$, the hyperbolic leaf function $sleafh_1(l)$ represents the hyperbolic function $sinh(l)$. With respect to an arbitrary basis n, the hyperbolic leaf function $sleafh_n(l)$ is closely related to the leaf function $sleaf_n(l)$.

In this paper, the hyperbolic leaf function $cleafh_n(l)$ is defined. This function is satisfied by the abovementioned ordinary differential equation and the following initial conditions:

$$\frac{d^2r(l)}{dl^2} = n \cdot r(l)^{2n-1} \quad n = 1,2,3,\cdots$$
$$r(0) = 1$$
$$\frac{dr(0)}{dl} = 0$$

Compared to the hyperbolic leaf function $sleafh_n(l)$, only the initial condition of the hyperbolic leaf function $cleafh_n(l)$ is different. In the case of the basis $n = 1$, the function represents the hyperbolic function $cosh(l)$. This function is closely related to other functions $cleaf_n(l)$, $sleaf_n(l)$, and $sleafh_n(l)$.

Keywords : Leaf function, Jacobi elliptic functions, Ordinary differential equation, Trigonometric function, Hyperbolic function, Square root of polynomial, Elliptic integral

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1. Introduction

In this paper, the hyperbolic leaf function $cleafh_n(l)$ is presented. This function is satisfied by the ordinary differential equation (ODE) and the following initial conditions:

$$\frac{d^2r(l)}{dl^2} = n \cdot r(l)^{2n-1} \quad n = 1,2,3,\cdots \tag{1}$$

$$r(0) = 1 \quad (or \quad r = 1, \quad l = 0) \tag{2}$$

$$\frac{dr(0)}{dl} = 0 \tag{3}$$

Compared to the hyperbolic leaf function $sleafh_n(l)$, only the initial condition of the hyperbolic leaf function $cleafh_n(l)$ is different. In the case of the basis $n = 1$, the function represents the hyperbolic function $cosh(l)$.

2. Definition of Hyperbolic Leaf Function $cleafh_n(l)$

In this section, we discuss about Eq. (1). The basis n represents the natural number $1, 2, 3, \cdots$ By multiplying dr/dl to both sides of Eq. (1), the following equation is obtained:

$$\frac{dr}{dl}\frac{d^2r}{dl^2} = nr^{2n-1}\frac{dr}{dl} \quad n = 1,2,3,\cdots \tag{4}$$

By integrating both sides of the above equation, the following equation is obtained:

$$\frac{1}{2}\left(\frac{dr}{dl}\right)^2 = \frac{1}{2}r^{2n} + C \quad n = 1,2,3,\cdots \tag{5}$$

C represents the constant of integration. C is determined by the initial conditions (Eqs. (2)–(3)). Therefore, the equation is as follows:

$$C = -\frac{1}{2} \tag{6}$$

Using the above results and Eq. (5), the following equation is obtained:

$$\frac{dr}{dl} = \pm\sqrt{r^{2n} - 1} \quad (r \geq 1) \tag{7}$$

where the variable r is satisfied by the inequality $r \geqq 1$. In the inequality $l \geqq 0$, the following equation is applied:

$$\frac{dr}{dl} = \sqrt{r^{2n} - 1} \quad (r \geq 1) \tag{8}$$

In the inequality $l < 0$, the following equation is applied:

$$\frac{dr}{dl} = -\sqrt{r^{2n} - 1} \quad (r \geq 1) \tag{9}$$

In the inequality $l \geqq 0$, Eq. (8) is integrated from 1 to r after the variables are separated.

$$\int_1^r dl = \int_1^r \frac{1}{\sqrt{t^{2n} - 1}}dt \quad (r \geq 1) \tag{10}$$

$$l(r) - l(1) = \int_1^r \frac{1}{\sqrt{t^{2n} - 1}}dt \quad (r \geq 1) \tag{11}$$

$$l(r) = l(1) + \int_1^r \frac{1}{\sqrt{t^{2n} - 1}}dt \quad (r \geq 1) \tag{12}$$

The symbol $l(1)(=l(r))$ becomes 0 by the initial condition of Eq. (2).

$$l(r) = \int_1^r \frac{1}{\sqrt{t^{2n} - 1}}dt \quad (r \geq 1) \tag{13}$$

To define the inverse function that satisfies the above equation, the following equation is formulated:

$$acleafh_n(r) = \int_1^r \frac{1}{\sqrt{t^{2n} - 1}}dt = l \tag{14}$$

In this paper, the prefix "*a*" of the hyperbolic leaf function $cleafh_n(l)$ represents the inverse function. Using the above equation, the following equation is obtained:

$$r = cleafh_n(l) \tag{15}$$

In the case of the basis $n = 1$, the following equation is obtained:

$$cleafh_1(l) = \cosh(l) \tag{16}$$

In the inequality $l < 0$, based on the Eq. (7), the following equation is defined:

$$l = -\int_1^r \frac{1}{\sqrt{t^{2n}-1}}dt \quad r \geq 1 \tag{17}$$

Using the above equation, the following equation is obtained:

$$r = cleafh_n(-l) \tag{18}$$

3. Graph of Hyperbolic Leaf Function: *cleafh$_n$(l)*

The hyperbolic leaf function is shown in Fig. 1.

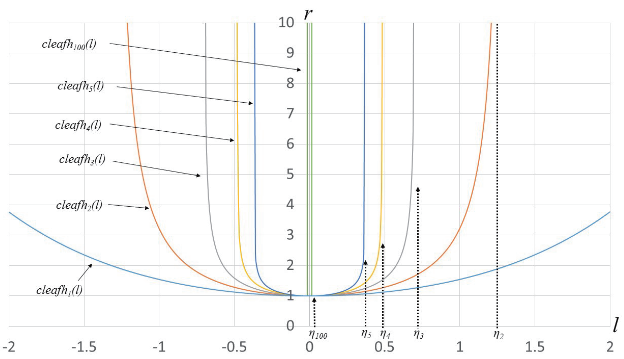

Fig. 1 Curve of the hyperbolic leaf function *cleafh$_n$(l)*

Variable r and variable l represent the vertical axis and the horizontal axis, respectively. The hyperbolic leaf function *cleafh$_n$(l)* is an even function. Therefore, it is obtained as follows:

$$cleafh_n(-l) = cleafh_n(l) \quad (n = 1,2,3,\cdots) \tag{19}$$

In the basis $n = 1$, the hyperbolic leaf function *cleafh$_1$(l)* represents the hyperbolic function *cosh(l)*. With respect to arbitrary basis n, the gradient of the function *cleafh$_n$(l)* becomes 0.0 at $l = 0.0$. It is based on the initial conditions (Eqs. (2) and (3)). As the basis n increases, the gradients of the curves become sharp. The hyperbolic leaf function *cleafh$_n$(l)* has the limit η_n except for the basis $n = 1$. We define the limit as follows:

$$\lim_{l \to \eta_n} cleafh_n(l) = \infty \quad (n = 2,3,\cdots) \tag{20}$$

The limit of the arbitrary basis n is obtained as follows:

$$\eta_n = \int_1^\infty \frac{1}{\sqrt{t^{2n}-1}}dt \ (=l) \quad (n = 2,3,\cdots) \tag{21}$$

The values of the limit η_n are summarized in Table 1.

Table 1 Limit η_n of variable l with respect to the hyperbolic leaf function *cleafh$_n$(l)* (All results have been rounded to no more than six significant figures)

Limit η_n	Value
η_1	N/A
η_2	1.31102
η_3	0.70109
η_4	0.48197
η_5	0.36790
η_{100}	0.01581

In the basis $n = 2$, the hyperbolic leaf function *cleafh$_n$(l)* become 0 if the following equation is satisfied:

$$l = \frac{\pi_2}{2} \tag{22}$$

where the constant π_2 is described in Ref. [2]. The following equation is obtained by substituting Eq. (22) in Eq. (15):

$$cleafh_2\left(\frac{\pi_2}{2}\right) = \infty \tag{23}$$

Based on the above equation, we can predict the limit by following equation:

$$l = \eta_2 = \frac{\pi_2}{2} \quad \left(= \int_1^\infty \frac{1}{\sqrt{t^4-1}}dt = \int_0^1 \frac{1}{\sqrt{1-t^4}}dt \right) \tag{24}$$

In the basis $n = 3$, we can predict the limit by the following equation:

$$l = \zeta_3 = 2\eta_3 \tag{25}$$

$$\zeta_3 = \int_0^{\infty} \frac{1}{\sqrt{1+t^6}} dt \tag{26}$$

$$2\eta_3 = 2\int_1^{\infty} \frac{1}{\sqrt{t^6-1}} dt \tag{27}$$

Based on the results of the numerical integration, we can find the above relation. The results of the limit are as follows:

$$\zeta_3 = 2\eta_3 = 1.4021821053254542611750190790502941354630222054239 \tag{28}$$

Using Eqs. (26) and (27), the limits ζ_3 and $2\times\eta_3$ are calculated by fifty digit numbers, respectively. The limit ζ_3 matches the limit $2\times\eta_3$ by fifty digit numbers.

Limit η_3 is also obtained by the following equation:

$$l = \eta_3 = \int_0^{\frac{1}{\sqrt{2}}} \frac{1}{\sqrt{1+t^6}} dt \tag{29}$$

Using the above equation, the following equation is obtained:

$$\begin{aligned} 2\eta_3 &= \int_0^{\infty} \frac{1}{\sqrt{1+t^6}} dt = \int_0^{\frac{1}{\sqrt{2}}} \frac{1}{\sqrt{1+t^6}} dt + \int_{\frac{1}{\sqrt{2}}}^{\infty} \frac{1}{\sqrt{1+t^6}} dt \\ &= \eta_3 + \int_{\frac{1}{\sqrt{2}}}^{\infty} \frac{1}{\sqrt{1+t^6}} dt \end{aligned} \tag{30}$$

Finally, the following equation is obtained:

$$\eta_3 = \int_{\frac{1}{\sqrt{2}}}^{\infty} \frac{1}{\sqrt{1+t^6}} dt \tag{31}$$

Eq. (29) represents the following equation:

$$sleafh_3(\eta_3) = \frac{1}{\sqrt{2}} \tag{32}$$

4. Extended Definition of Hyperbolic Leaf Function *cleafh$_n$(l)*

With respect to an arbitrary variable l, the value of the leaf function $cleaf_n(l)$ can be obtained. On the other hand, except for the basis $n = 1$, the hyperbolic leaf function $cleafh_n(l)$ can only be obtained within the domain of the variable:

$$-\zeta_n < l < \zeta_n \quad (n = 2,3,4,\cdots) \tag{33}$$

The function is not supported for arbitrary variable l. Therefore, the hyperbolic leaf function is redefined as the multivalued function, so that the arbitrary variable r can correspond to the arbitrary variable l

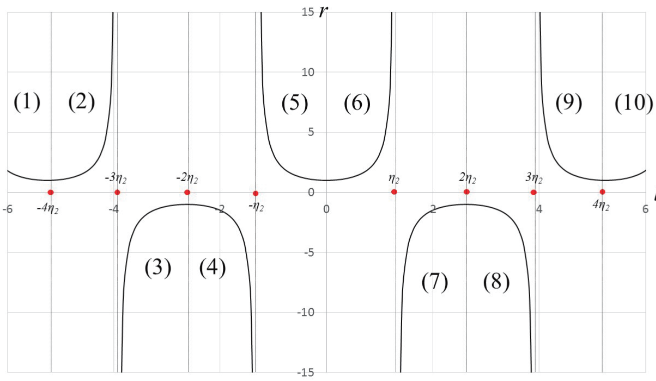

Fig. 2 Curve of the hyperbolic leaf function $r = cleafh_2(l)$

In the case of the basis $n = 2$, the curve of the hyperbolic leaf function is shown in Fig. 2. Numbers (1)–(10) represent the domain. By separating domains (1)–(10) with respect to the variable l, the relation between variable r and variable l is redefined. First, in the domain (1), gradient dl/dr becomes negative.

$$\frac{dl}{dr} = -\frac{1}{\sqrt{r^4-1}} \quad (r \geq 1, \quad -5\eta_2 < l \leq -4\eta_2) \tag{34}$$

The initial condition in the domain (1) is defined as the initial condition: $l(1) = -4\eta_2$. The above equation is integrated from the number 1 to the variable r.

$$\begin{aligned} l(r) = l(1) - \int_1^r \frac{1}{\sqrt{t^4-1}} dt &= -4\eta_2 - \int_1^r \frac{1}{\sqrt{t^4-1}} dt \\ (r \geq 1, \quad -5\eta_2 < l \leq -4\eta_2) \end{aligned} \tag{35}$$

In the domain (2), gradient *dl/dr* becomes positive.

$$\frac{dl}{dr}=\frac{1}{\sqrt{r^4-1}}\quad \left(r\geq 1,\quad -4\eta_2\leq l<-3\eta_2\right) \tag{36}$$

The initial condition in domain (2) is defined as the initial condition: $l(1) = -4\eta_2$. The above equation is integrated from the number *1* to the variable *r*.

$$l(r)=l(1)+\int_1^r\frac{1}{\sqrt{t^4-1}}dt=-4\eta_2+\int_1^r\frac{1}{\sqrt{t^4-1}}dt \quad \left(r\geq 1,\quad -4\eta_2\leq l<-3\eta_2\right) \tag{37}$$

In the domain (3), gradient *dl/dr* becomes positive.

$$\frac{dl}{dr}=\frac{1}{\sqrt{r^4-1}}\quad \left(r\leq -1,\quad -3\eta_2< l\leq -2\eta_2\right) \tag{38}$$

The initial condition in the domain (3) is defined as the initial condition: $l(-1) = -2\eta_2$. The above equation is integrated from the number *-1* to the variable *r*.

$$l(r)=l(-1)+\int_{-1}^r\frac{1}{\sqrt{t^4-1}}dt=-2\eta_2-\int_r^{-1}\frac{1}{\sqrt{t^4-1}}dt \quad \left(r\leq -1,\quad -3\eta_2< l\leq -2\eta_2\right) \tag{39}$$

In the domain (4), gradient *dl/dr* becomes negative.

$$\frac{dl}{dr}=-\frac{1}{\sqrt{r^4-1}}\quad \left(r\leq -1,\quad -2\eta_2\leq l<-\eta_2\right) \tag{40}$$

The initial condition in the domain (4) is defined as the initial condition: $l(-1) = -2\eta_2$. The above equation is integrated from the number *-1* to the variable *r*.

$$l(r)=l(-1)+\int_{-1}^r\left(-\frac{1}{\sqrt{t^4-1}}\right)dt=-2\eta_2+\int_r^{-1}\frac{1}{\sqrt{t^4-1}}dt \quad \left(r\leq -1,\quad -2\eta_2\leq l<-\eta_2\right) \tag{41}$$

In the domain (5), gradient *dl/dr* becomes negative.

$$\frac{dl}{dr}=-\frac{1}{\sqrt{r^4-1}}\quad \left(r\geq 1,\quad -\eta_2< l\leq 0\right) \tag{42}$$

The initial condition in the domain (5) is defined as the initial condition: $l(1) = 0$. The above equation is integrated from the number *1* to the variable *r*.

$$l(r)=l(1)+\int_1^r\left(-\frac{1}{\sqrt{t^4-1}}\right)dt=\int_r^1\frac{1}{\sqrt{t^4-1}}dt \quad \left(r\geq 1,\quad -\eta_2< l\leq 0\right) \tag{43}$$

In the domain (6), gradient *dl/dr* becomes positive.

$$\frac{dl}{dr}=\frac{1}{\sqrt{r^4-1}}\quad \left(r\geq 1,\quad 0\leq l<\eta_2\right) \tag{44}$$

The initial condition in domain (6) is defined as the initial condition: $l(1) = 0$. The above equation is integrated from the number *1* to the variable *r*.

$$l(r)=l(1)+\int_1^r\frac{1}{\sqrt{t^4-1}}dt=\int_1^r\frac{1}{\sqrt{t^4-1}}dt \quad \left(r\geq 1,\quad 0\leq l<\eta_2\right) \tag{45}$$

In the domain (7), gradient *dl/dr* becomes positive.

$$\frac{dl}{dr}=\frac{1}{\sqrt{r^4-1}}\quad \left(r\geq 1,\quad \eta_2< l\leq 2\eta_2\right) \tag{46}$$

The initial condition in the domain (7) is defined as the initial condition: $l(-1) = 2\eta_2$. The above equation is integrated from the number *-1* to the variable *r*.

$$l(r)=l(-1)+\int_{-1}^r\frac{1}{\sqrt{t^4-1}}dt=2\eta_2-\int_r^{-1}\frac{1}{\sqrt{t^4-1}}dt \quad \left(r\leq -1,\quad \eta_2< l\leq 2\eta_2\right) \tag{47}$$

In the domain (8), gradient *dl/dr* becomes negative.

$$\frac{dl}{dr}=-\frac{1}{\sqrt{r^4-1}}\quad \left(r\leq -1,\quad 2\eta_2\leq l<3\eta_2\right) \tag{48}$$

The initial condition in the domain (8) is defined as the initial condition: $l(-1) = 2\eta_2$. The above equation is integrated from the number *-1* to the variable *r*.

$$l(r)=l(-1)+\int_{-1}^r\left(-\frac{1}{\sqrt{t^4-1}}\right)dt=2\eta_2+\int_r^{-1}\frac{1}{\sqrt{t^4-1}}dt \quad \left(r\leq -1,\quad 2\eta_2\leq l<3\eta_2\right) \tag{49}$$

In the domain (9), gradient dl/dr becomes negative.

$$\frac{dl}{dr}=-\frac{1}{\sqrt{r^4-1}} \quad (r\geq 1, \quad 3\eta_2<l\leq 4\eta_2) \tag{50}$$

The initial condition in the domain (9) is defined as the initial condition: $l(1)=4\eta_2$. The above equation is integrated from the number 1 to the variable r.

$$l(r)=l(1)+\int_1^r\left(-\frac{1}{\sqrt{t^4-1}}\right)dt=4\eta_2-\int_1^r\frac{1}{\sqrt{t^4-1}}dt \tag{51}$$
$$(r\geq 1, \quad 3\eta_2<l\leq 4\eta_2)$$

In the domain (10), gradient dl/dr becomes positive.

$$\frac{dl}{dr}=\frac{1}{\sqrt{r^4-1}} \quad (r\geq 1, \quad 4\eta_2\leq l<5\eta_2) \tag{52}$$

The initial condition in the domain (10) is defined as the initial condition: $l(1)=4\eta_2$. The above equation is integrated from the number 1 to the variable r.

$$l(r)=l(1)+\int_1^r\frac{1}{\sqrt{t^4-1}}dt=4\eta_2+\int_1^r\frac{1}{\sqrt{t^4-1}}dt \tag{53}$$
$$(r\geq 1, \quad 4\eta_2\leq l<5\eta_2)$$

With respect to arbitrary variable l, the relation between variable r and variable l is summarized in Table 2. The symbols n and m represent the basis and the integer number, respectively.

In the case of the basis $n=2, 3, 4, 5,$ and 100, the graphs are shown from Fig. 2 to Fig. 6, respectively. The vertical axis and the horizontal axis represent variable r and variable l, respectively. Alternatively, both curves of a downward convex and an upward convex exist. In the case of a small basis n, the curve tends to be smooth and rounded. In the case of a large basis n, the curve tends to be sharp and angulated.

Table 2 Relation between variable r and variable l (Based on the hyperbolic leaf function $cleafh_2(l)$) (except for $n=1$)

Domain	Domain of $cleafh_2(l)$	Initial condition	Calculation formula and derivative
(1)	$-5\eta_2<l\leqq-4\eta_2$ $r\geqq 1$	$l=-4\eta_2$ $r=1$	$l(r)=-4\eta_2-\int_1^r\frac{1}{\sqrt{t^4-1}}dt$ $\frac{dl}{dr}=-\frac{1}{\sqrt{r^4-1}}$
(2)	$-4\eta_2\leqq l<-3\eta_2$ $r\geqq 1$	$l=-4\eta_2$ $r=1$	$l(r)=-4\eta_2+\int_1^r\frac{1}{\sqrt{t^4-1}}dt$ $\frac{dl}{dr}=\frac{1}{\sqrt{r^4-1}}$
(3)	$-3\eta_2<l\leqq-2\eta_2$ $r\leqq -1$	$l=-2\eta_2$ $r=-1$	$l(r)=-2\eta_2-\int_r^{-1}\frac{1}{\sqrt{t^4-1}}dt$ $\frac{dl}{dr}=\frac{1}{\sqrt{r^4-1}}$
(4)	$-2\eta_2\leqq l<-\eta_2$ $r\leqq -1$	$l=-2\eta_2$ $r=-1$	$l(r)=-2\eta_2+\int_r^{-1}\frac{1}{\sqrt{t^4-1}}dt$ $\frac{dl}{dr}=-\frac{1}{\sqrt{r^4-1}}$
(5)	$-\eta_2<l\leqq 0$ $r\geqq 1$	$l=0$ $r=1$	$l(r)=-\int_1^r\frac{1}{\sqrt{t^4-1}}dt$ $\frac{dl}{dr}=-\frac{1}{\sqrt{r^4-1}}$
(6)	$0\leqq l<\eta_2$ $r\geqq 1$	$l=0$ $r=1$	$l(r)=\int_1^r\frac{1}{\sqrt{t^4-1}}dt$ $\frac{dl}{dr}=\frac{1}{\sqrt{r^4-1}}$
(7)	$\eta_2<l\leqq 2\eta_2$ $r\leqq -1$	$l=2\eta_2$ $r=-1$	$l(r)=2\eta_2-\int_r^{-1}\frac{1}{\sqrt{t^4-1}}dt$ $\frac{dl}{dr}=\frac{1}{\sqrt{r^4-1}}$
(8)	$2\eta_2\leqq l<3\eta_2$ $r\leqq -1$	$l=2\eta_2$ $r=-1$	$l(r)=2\eta_2+\int_r^{-1}\frac{1}{\sqrt{t^4-1}}dt$ $\frac{dl}{dr}=-\frac{1}{\sqrt{r^4-1}}$
(9)	$3\eta_2<l\leqq 4\eta_2$ $r\geqq 1$	$l=4\eta_2$ $r=1$	$l(r)=4\eta_2-\int_1^r\frac{1}{\sqrt{t^4-1}}dt$ $\frac{dl}{dr}=-\frac{1}{\sqrt{r^4-1}}$
(10)	$4\eta_2\leqq l<5\eta_2$ $r\geqq 1$	$l=4\eta_2$ $r=1$	$l(r)=4\eta_2+\int_1^r\frac{1}{\sqrt{t^4-1}}dt$ $\frac{dl}{dr}=\frac{1}{\sqrt{r^4-1}}$

Table 3 Relation between variable r and variable l (Based on the hyperbolic leaf function $cleafh_n(l)$) (except for $n=1$)

Domain of $cleafh_n(l)$	Initial condition	Calculation formula and derivative
$(4m-1)\eta_n < l \leqq 4m\eta_n$ $r \geqq 1$	$l=4m\eta_n$ $r=1$	$l(r)=4m\eta_n-\int_1^r \frac{1}{\sqrt{t^{2n}-1}}dt$ $\frac{dl}{dr}=-\frac{1}{\sqrt{r^{2n}-1}}$
$4m\eta_n \leqq l < (4m+1)\eta_n$ $r \geqq 1$	$l=4m\eta_n$ $r=1$	$l(r)=4m\eta_n+\int_1^r \frac{1}{\sqrt{t^{2n}-1}}dt$ $\frac{dl}{dr}=\frac{1}{\sqrt{r^{2n}-1}}$
$(4m+1)\eta_n < l \leqq (4m+2)\eta_n$ $r \leqq -1$	$l=(4m+2)\eta_n$ $r=-1$	$l(r)=(4m+2)\eta_n-\int_r^{-1} \frac{1}{\sqrt{t^{2n}-1}}dt$ $\frac{dl}{dr}=\frac{1}{\sqrt{r^{2n}-1}}$
$(4m+2)\eta_n < l \leqq (4m+3)\eta_n$ $r \leqq -1$	$l=(4m+2)\eta_n$ $r=-1$	$l(r)=(4m+2)\eta_n+\int_r^{-1} \frac{1}{\sqrt{t^{2n}-1}}dt$ $\frac{dl}{dr}=-\frac{1}{\sqrt{r^{2n}-1}}$

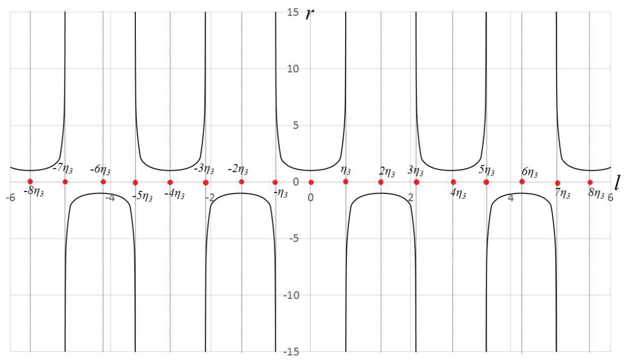

Fig. 3 Curve of the hyperbolic leaf function $r = cleafh_3(l)$

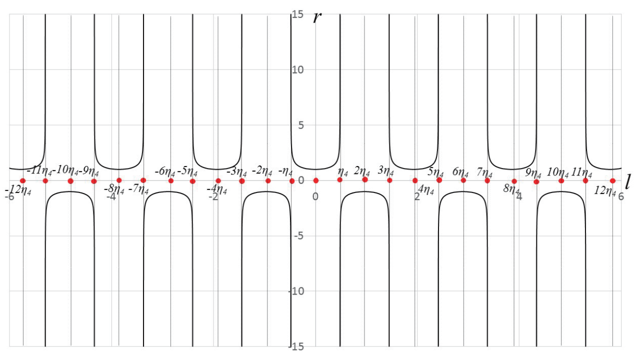

Fig. 4 Curve of the hyperbolic leaf function $r = cleafh_4(l)$

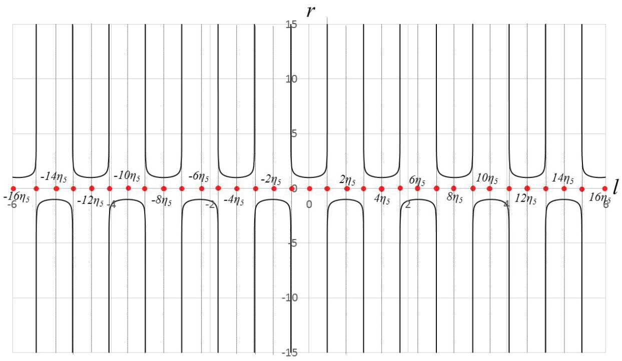

Fig. 5 Curve of hyperbolic leaf function $r = cleafh_5(l)$

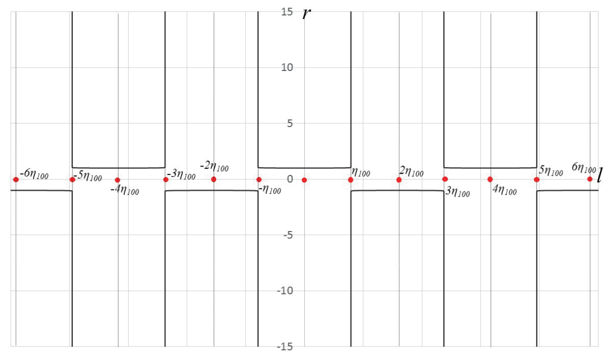

Fig. 6 Curve of the hyperbolic leaf function $r = cleafh_{100}(l)$

5. Relation Between Hyperbolic Leaf Function $cleafh_n(l)$ and Other Function

In the case of the basis $n = 1$, the relation between the functions $cleafh_1(l)$ and $sleafh_1(l)$ is obtained as follows:

$$(cleafh_1(l))^2-(sleafh_1(l))^2=1 \tag{54}$$

The above equation represents the relation between the hyperbolic function $cosh(l)$ and the hyperbolic function $sinh(l)$.

In the case of the basis $n = 2$, the following equations are obtained:

$$cleaf_2(l)\cdot cleafh_2(l)=1 \tag{55}$$

$$cleafh_2(\sqrt{2}l)=\frac{1+(sleafh_2(l))^2}{1-(sleafh_2(l))^2} \tag{56}$$

For more information, see appendix B and C. In the case of the basis $n = 3$, the following equations are obtained:

$$\left(cleafh_3(l)\right)^2-\left(sleafh_3(l)\right)^2-2\left(cleafh_3(l)\right)^2\left(sleafh_3(l)\right)^2=1$$
$$(4m-1)\eta_3<l<(4m+1)\eta_3 \tag{57}$$

The functions $cleafh_3(l)$ and $sleafh_3(l)$ are defined as the multivalued function with periodicity η_3 and periodicity ζ_3, respectively. Periodicity η_3 does not match periodicity ζ_3. Periodicity η_3 of the function $cleafh_3(l)$ is shorter than periodicity ζ_3 of the function $sleafh_3(l)$. The above equation is satisfied in the partial domain: ($(4m-1)\eta_3<l<(4m+1)\eta_3$, m: integer number).

6. Addition Theorem of Leaf Function

The addition theorem of the hyperbolic leaf function is described in this section. In the case of the basis $n=1$, the following equation is obtained:

$$sleafh_1(l_1+l_2)=sleafh_1(l_1)\cdot cleafh_1(l_2)+cleafh_1(l_1)\cdot sleafh_1(l_2) \tag{58}$$

$$cleafh_1(l_1+l_2)=cleafh_1(l_1)\cdot cleafh_1(l_2)+sleafh_1(l_1)\cdot sleafh_1(l_2) \tag{59}$$

These equations represent the relation between the hyperbolic function $sinh(l)$ and hyperbolic function $cosh(l)$. In the case of the basis $n=2$, the following equation is obtained:

$$sleafh_2(l_1+l_2)=\frac{sleafh_2(l_1)\sqrt{1+\left(sleafh_2(l_2)\right)^4}+sleafh_2(l_2)\sqrt{1+\left(sleafh_2(l_1)\right)^4}}{1-\left(sleafh_2(l_1)\right)^2\left(sleafh_2(l_2)\right)^2} \tag{60}$$

$$cleafh_2(l_1+l_2)=\frac{2cleafh_2(l_1)cleafh_2(l_2)+cleafh_2'(l_1)cleafh_2'(l_2)}{1+\left(cleafh_2(l_1)\right)^2+\left(cleafh_2(l_2)\right)^2-\left(cleafh_2(l_1)\right)^2\left(cleafh_2(l_2)\right)^2} \tag{61}$$

In the above equation, the superscript prime of the hyperbolic leaf function represents the derivative with respect to variable l. Based on the data from table 3, the sign (plus or minus) of the derivative is decided. In the case of the domain: $4m\eta_2 \leqq l_1 \leqq (4m+2)\eta_2$ and $4m\eta_2 \leqq l_2 \leqq (4m+2)\eta_2$ (m: integer), the following equation is obtained:

$$cleafh_2'(l_1)=\sqrt{\left(cleafh_2(l_1)\right)^4-1} \tag{62}$$

$$cleafh_2'(l_2)=\sqrt{\left(cleafh_2(l_2)\right)^4-1} \tag{63}$$

$$cleafh_2(l_1+l_2)=\frac{2cleafh_2(l_1)cleafh_2(l_2)+\sqrt{\left(cleafh_2(l_1)\right)^4-1}\sqrt{\left(cleafh_2(l_2)\right)^4-1}}{1+\left(cleafh_2(l_1)\right)^2+\left(cleafh_2(l_2)\right)^2-\left(cleafh_2(l_1)\right)^2\left(cleafh_2(l_2)\right)^2} \tag{64}$$

In the case of the domain: $4m\eta_2 \leqq l_1 \leqq (4m+2)\eta_2$ and $(4m+2)\eta_2 \leqq l_2 \leqq (4m+4)\eta_2$, the following equation is obtained:

$$cleafh_2'(l_1)=\sqrt{\left(cleafh_2(l_1)\right)^4-1} \tag{65}$$

$$cleafh_2'(l_2)=-\sqrt{\left(cleafh_2(l_2)\right)^4-1} \tag{66}$$

$$cleafh_2(l_1+l_2)=\frac{2cleafh_2(l_1)cleafh_2(l_2)-\sqrt{\left(cleafh_2(l_1)\right)^4-1}\sqrt{\left(cleafh_2(l_2)\right)^4-1}}{1+\left(cleafh_2(l_1)\right)^2+\left(cleafh_2(l_2)\right)^2-\left(cleafh_2(l_1)\right)^2\left(cleafh_2(l_2)\right)^2} \tag{67}$$

In the case of the domain: $(4m+2)\eta_2 \leqq l_1 \leqq (4m+4)\eta_2$ and $4m\eta_2 \leqq l_2 \leqq (4m+2)\eta_2$, the following equation is obtained:

$$cleafh_2'(l_1)=-\sqrt{\left(cleafh_2(l_1)\right)^4-1} \tag{68}$$

$$cleafh_2'(l_2)=\sqrt{\left(cleafh_2(l_2)\right)^4-1} \tag{69}$$

$$cleafh_2(l_1+l_2)=\frac{2cleafh_2(l_1)cleafh_2(l_2)-\sqrt{\left(cleafh_2(l_1)\right)^4-1}\sqrt{\left(cleafh_2(l_2)\right)^4-1}}{1+\left(cleafh_2(l_1)\right)^2+\left(cleafh_2(l_2)\right)^2-\left(cleafh_2(l_1)\right)^2\left(cleafh_2(l_2)\right)^2} \tag{70}$$

In the case of the domain: $(4m+2)\eta_2 \leqq l_1 \leqq (4m+4)\eta_2$ and $(4m+2)\eta_2 \leqq l_2 \leqq (4m+4)\eta_2$, the following equation is obtained:

$$cleafh_2'(l_1)=-\sqrt{\left(cleafh_2(l_1)\right)^4-1} \tag{71}$$

$$cleafh_2'(l_2) = -\sqrt{(cleafh_2(l_2))^4 - 1} \tag{72}$$

$$cleafh_2(l_1+l_2) = \frac{2cleafh_2(l_1)cleafh_2(l_2) + \sqrt{(cleafh_2(l_1))^4 - 1}\sqrt{(cleafh_2(l_2))^4 - 1}}{1 + (cleafh_2(l_1))^2 + (cleafh_2(l_2))^2 - (cleafh_2(l_1))^2(cleafh_2(l_2))^2} \tag{73}$$

7. Maclaurin Series of Hyperbolic Leaf Function

In this section, the Maclaurin series is applied to the hyperbolic leaf function. In the case of $n = 2$, the function $cleafh_2(l)$ is expanded as follows:

$$cleafh_2(l) = 1 + l^2 + \frac{1}{2}l^4 + \frac{3}{10}l^6 + \frac{7}{40}l^8 + O(l^{10}) \tag{74}$$

For more information, see Appendix A. Symbol $O(l^{10})$ represents the Landau symbol (the big O notation).

$$\lim_{l \to 0} \frac{O(l^{10})}{l^{10}} = \frac{cleafh_2(l) - \left(1 + l^2 + \frac{1}{2}l^4 + \frac{3}{10}l^6 + \frac{7}{40}l^8\right)}{l^{10}} = \frac{61}{600} \tag{75}$$

Subsequently, in the case of $n = 3$, the hyperbolic leaf function $cleafh_3(l)$ can be expanded by the Maclaurin series as follows:

$$cleafh_3(l) = 1 + \frac{3}{2}l^2 + \frac{15}{8}l^4 + \frac{51}{16}l^6 + \frac{5085}{896}l^8 + O(l^{10}) \tag{76}$$

In the case of $n = 4$, the hyperbolic leaf function $cleafh_4(l)$ can be expanded by the Maclaurin series as follows:

$$cleafh_4(l) = 1 + 2l^2 + \frac{14}{3}l^4 + \frac{140}{9}l^6 + \frac{502}{9}l^8 + O(l^{10}) \tag{77}$$

In the case of $n = 5$, the hyperbolic leaf function $cleafh_5(l)$ can be expanded by the Maclaurin series as follows:

$$cleafh_5(l) = 1 + \frac{5}{2}l^2 + \frac{75}{8}l^4 + \frac{825}{16}l^6 + \frac{277125}{896}l^8 + O(l^{10}) \tag{78}$$

8. Relation Between Leaf Function *cleaf$_n$(l)* and Hyperbolic Leaf Function *cleafh$_n$(l)*

Using complex numbers, the relation between leaf function $cleaf_n(l)$ and hyperbolic leaf function $cleafh_n(l)$ is shown. The complex variable $i \cdot l$ is substituted for the variables l in the Maclaurin series of both functions $cleaf_n(l)$ (See Ref.[2]) and $cleafh_n(l)$. Symbol i represents the imaginary number. In the case of the basis $n = 1$, the function $cleaf_1(l)$ and the function $cleafh_1(l)$ represent the function $cos(l)$ and the function $cosh(l)$, respectively. Therefore, the following equation is obtained:

$$cleaf_1(i \cdot l) = cleafh_1(l) \quad (\cos(i \cdot l) = \cosh(l)) \tag{79}$$

In the case of the basis $n = 2$, the following equation is obtained:

$$\begin{aligned} cleafh_2(i \cdot l) &= 1 + (i \cdot l)^2 + \frac{1}{2}(i \cdot l)^4 + \frac{3}{10}(i \cdot l)^6 + \frac{7}{40}(i \cdot l)^8 + O((i \cdot l)^{10}) \\ &= 1 + i^2 \cdot l^2 + \frac{1}{2}i^4 \cdot l^4 + \frac{3}{10}i^6 \cdot l^6 + \frac{7}{40}i^8 \cdot l^8 + O(i^{10} \cdot l^{10}) \\ &= 1 - l^2 + \frac{1}{2}l^4 - \frac{3}{10}l^6 + \frac{7}{40}l^8 - O(l^{10}) = cleaf_2(l) \end{aligned} \tag{80}$$

In the case of the basis $n = 3$, the following equation is obtained:

$$\begin{aligned} cleafh_3(i \cdot l) &= 1 + \frac{3}{2}i^2 \cdot l^2 + \frac{15}{8}i^4 \cdot l^4 + \frac{51}{16}i^6 \cdot l^6 + \frac{5085}{896}i^8 \cdot l^8 + O(i^{10} \cdot l^{10}) \\ &= 1 - \frac{3}{2}l^2 + \frac{15}{8}l^4 - \frac{51}{16}l^6 + \frac{5085}{896}l^8 - O(l^{10}) = cleaf_3(l) \end{aligned} \tag{81}$$

In the case of the basis $n = 4$, the following equation is obtained:

$$\begin{aligned} cleafh_4(i \cdot l) &= 1 + 2i^2 \cdot l^2 + \frac{14}{3}i^4 \cdot l^4 + \frac{140}{9}i^6 \cdot l^6 + \frac{502}{9}i^8 \cdot l^8 + O(i^{10} \cdot l^{10}) \\ &= 1 - 2l^2 + \frac{14}{3}l^4 - \frac{140}{9}l^6 + \frac{502}{9}l^8 - O(l^{10}) = cleaf_4(l) \end{aligned} \tag{82}$$

In the case of the basis $n = 5$, the following equation is obtained:

$$
\begin{aligned}
& cleafh_5(i \cdot l) \\
& = 1 + \frac{5}{2} i^2 \cdot l^2 + \frac{75}{8} i^4 \cdot l^4 + \frac{825}{16} i^6 \cdot l^6 + \frac{277125}{896} i^8 \cdot l^8 + O\left(i^{10} \cdot l^{10}\right) \\
& = 1 - \frac{5}{2} l^2 + \frac{75}{8} l^4 - \frac{825}{16} l^6 + \frac{277125}{896} l^8 - O\left(l^{10}\right) = cleaf_5(l)
\end{aligned}
\tag{83}
$$

Based on the above results, the following equation can be predicted:

$$
cleafh_n(i \cdot l) = cleaf_n(l) \tag{84}
$$

9. Conclusion

In this report, the hyperbolic leaf function: *cleafhn(l)* is defined. The second derivative of the function is equal to the positive operator of the function with power *2n - 1* (*n*: natural number). The conclusions are summarized as follows:

・In the case of $n = 1$, the function: $cleafh_1(l)$ represents the hyperbolic function: *cosh(l)*.
・As number *n* increases, the smooth curve of the function tends to be a convex or concave curve.
・In the case of the condition $n \geqq 2$, the function $cleafh_n(l)$ has the limit with respect to variable *l*.
・The equation between the hyperbolic leaf function and the leaf function is formulated by using the imaginary number.

References

[1] Kazunori Shinohara, Special function: Leaf Function $r=sleaf_n(l)$ (First report), Bulletin of Daido University, **51**(2015), pp. 23–38.
[2] Kazunori Shinohara, Special function: Leaf Function $r=cleaf_n(l)$ (Second report), Bulletin of Daido University, **51**(2015), pp. 39–68.
[3] Kazunori Shinohara, Special function: Hyperbolic Leaf Function $r=sleafh_n(l)$ (First report), Bulletin of Daido University, **52**(2016), pp. 65 – 80.

Appendix A

In the case of $n = 2, 3, 4, 5$, the derivative and the Maclaurin series of the hyperbolic leaf function are described in this section. First, the hyperbolic leaf function: $cleafh_2(l)$ is expanded as the Maclaurin series. The first derivative of the hyperbolic leaf function $cleafh_2(l)$ is as follows:

$$
\frac{d}{dl} cleafh_2(l) = \sqrt{\left(cleafh_2(l)\right)^4 - 1} \tag{A1}
$$

The second derivative of the hyperbolic leaf function $cleafh_2(l)$ is as follows:

$$
\frac{d^2}{dl^2} cleafh_2(l) = 2 \cdot \left(cleafh_2(l)\right)^3 \tag{A2}
$$

The third derivative of the hyperbolic leaf function $cleafh_2(l)$ is as follows:

$$
\frac{d^3}{dl^3} cleafh_2(l) = 6 \cdot \left(cleafh_2(l)\right)^2 \cdot \sqrt{\left(cleafh_2(l)\right)^4 - 1} \tag{A3}
$$

The fourth derivative of the hyperbolic leaf function $cleafh_2(l)$ is as follows:

$$
\frac{d^4}{dl^4} cleafh_2(l) = 12 \cdot cleafh_2(l) \cdot \left(2\left(cleafh_2(l)\right)^4 - 1\right) \tag{A4}
$$

The fifth derivative of the hyperbolic leaf function $cleafh_2(l)$ is as follows:

$$
\frac{d^5}{dl^5} cleafh_2(l) = 12 \cdot \left(10 \cdot \left(cleafh_2(l)\right)^4 - 1\right) \sqrt{\left(cleafh_2(l)\right)^4 - 1} \tag{A5}
$$

The sixth derivative of the hyperbolic leaf function $cleafh_2(l)$ is as follows:

$$
\frac{d^6}{dl^6} cleafh_2(l) = 72\left(cleafh_2(l)\right)^3 \left(10\left(cleafh_2(l)\right)^4 - 7\right) \tag{A6}
$$

The seventh derivative of the hyperbolic leaf function $cleafh_2(l)$ is as follows:

$$\frac{d^7}{dl^7}cleafh_2(l) = 504(cleafh_2(l))^2(10(cleafh_2(l))^4-3)\sqrt{(cleafh_2(l))^4-1} \tag{A7}$$

The eighth derivative of the hyperbolic leaf function *cleafh₂(l)* is as follows:

$$\frac{d^8}{dl^8}cleafh_2(l) = 1008\,cleafh_2(l)\left(3-36(cleafh_2(l))^4+40(cleafh_2(l))^8\right) \tag{A8}$$

The ninth derivative of the hyperbolic leaf function $cleafh_2(l)$ is as follows:

$$\frac{d^9}{dl^9}cleafh_2(l) = 3024\left(1-60(cleafh_2(l))^4+120(cleafh_2(l))^8\right)\sqrt{(cleafh_2(l))^4-1} \tag{A9}$$

The tenth derivative of the hyperbolic leaf function $cleafh_2(l)$ is as follows:

$$\frac{d^{10}}{dl^{10}}cleafh_2(l) = 6048(cleafh_2(l))^3\left(121-660(cleafh_2(l))^4+600(cleafh_2(l))^8\right) \tag{A10}$$

The eleventh derivative of the hyperbolic leaf function $cleafh_2(l)$ is as follows:

$$\frac{d^{11}}{dl^{11}}cleafh_2(l) = 199584\,(cleafh_2(l))^2\cdot \left(11-140(cleafh_2(l))^4+200(cleafh_2(l))^8\right)\sqrt{(cleafh_2(l))^4-1} \tag{A11}$$

The twelfth derivative of the hyperbolic leaf function $cleafh_2(l)$ is as follows:

$$\frac{d^{12}}{dl^{12}}cleafh_2(l) = 399168\,cleafh_2(l)\cdot \left(-11+2(cleafh_2(l))^4\left(221-780(cleafh_2(l))^4+600(cleafh_2(l))^8\right)\right) \tag{A12}$$

The thirteenth derivative of the hyperbolic leaf function: $sleafh_2(l)$ is as follows

$$\frac{d^{13}}{dl^{13}}cleafh_2(l) = 399168\sqrt{(cleafh_2(l))^4-1} \cdot\left\{-11+130(cleafh_2(l))^4\left(17-108(cleafh_2(l))^4+120(cleafh_2(l))^8\right)\right\} \tag{A13}$$

Using the derivatives from Eqs. (A1)–(A13), the Maclaurin series of the hyperbolic leaf function: $cleafh_2(l)$ is formulated as follows:

$$\begin{aligned} cleafh_2(l) &= cleafh_2(0)+\frac{1}{1!}\left(\frac{d}{dl}cleafh_2(0)\right)l+\frac{1}{2!}\left(\frac{d^2}{dl^2}cleafh_2(0)\right)l^2 \\ &+\frac{1}{3!}\left(\frac{d^3}{dl^3}cleafh_2(0)\right)l^3+\cdots+\frac{1}{9!}\left(\frac{d^9}{dl^9}cleafh_2(0)\right)l^9+\cdots \\ &= 1+\frac{2}{2!}l+\frac{12}{4!}l^4+\frac{216}{6!}l^6+\frac{7056}{8!}l^8+O(l^{10}) \\ &= 1+l^2+\frac{1}{2}l^4+\frac{3}{10}l^6+\frac{7}{40}l^8+O(l^{10}) \end{aligned} \tag{A14}$$

Symbol O represents the Landau symbol. Using the above equation, the second derivative with respect to variable l is obtained as follows:

$$\frac{d^2}{dl^2}cleafh_2(l) = 2+6l^2+9l^4+\frac{49}{5}l^6+O(l^8) \tag{A15}$$

Using Eq. (A14), the following equation is obtained:

$$\begin{aligned} 2\cdot(cleafh_2(l))^3 &= 2\cdot\left(1+l^2+\frac{1}{2}l^4+\frac{3}{10}l^6+\frac{7}{40}l^8+O(l^{10})\right)^3 \\ &= 2+6l^2+9l^4+\frac{49}{5}l^6+O(l^{18}) \end{aligned} \tag{A16}$$

Eq. (A15) is equal to Eq. (A16). Therefore, the hyperbolic leaf function: $cleafh_2(l)$ satisfies Eq. (1). Subsequently, in the case of the basis $n = 3$, the Maclaurin series is applied to the hyperbolic leaf function: $cleafh_3(l)$. The first derivative of the hyperbolic leaf function: $cleafh_3(l)$ is as follows:

$$\frac{d}{dl}cleafh_3(l) = \sqrt{(cleafh_3(l))^6-1} \tag{A17}$$

The second derivative of the hyperbolic leaf function

$cleafh_3(l)$ is as follows:

$$\frac{d^2}{dl^2} cleafh_3(l) = 3 \cdot cleafh_3^5(l) \tag{A18}$$

The third derivative of the hyperbolic leaf function $cleafh_3(l)$ is as follows:

$$\frac{d^3}{dl^3} cleafh_3(l) = 15 \cdot \left(cleafh_3(l)\right)^4 \cdot \sqrt{\left(cleafh_3(l)\right)^6 - 1} \tag{A19}$$

The fourth derivative of the hyperbolic leaf function $cleafh_3(l)$ is as follows:

$$\frac{d^4}{dl^4} cleafh_3(l) = 15 \cdot \left(cleafh_3(l)\right)^3 \cdot \left(7\left(cleafh_3(l)\right)^6 - 4\right) \tag{A20}$$

The fifth derivative of the hyperbolic leaf function $cleafh_3(l)$ is as follows:

$$\begin{aligned} &\frac{d^5}{dl^5} cleafh_3(l) \\ &= 45\left(cleafh_3(l)\right)^2 \left(21\left(cleafh_3(l)\right)^6 - 4\right)\sqrt{\left(cleafh_3(l)\right)^6 - 1} \end{aligned} \tag{A21}$$

The sixth derivative of the hyperbolic leaf function $cleafh_3(l)$ is as follows:

$$\begin{aligned} &\frac{d^6}{dl^6} cleafh_3(l) \\ &= 45 cleafh_3(l)\left(8 - 188\left(cleafh_3(l)\right)^6 + 231\left(cleafh_3(l)\right)^{12}\right) \end{aligned} \tag{A22}$$

The seventh derivative of the hyperbolic leaf function $cleafh_3(l)$ is as follows:

$$\begin{aligned} &\frac{d^7}{dl^7} cleafh_3(l) = 45\sqrt{\left(cleafh_3(l)\right)^6 - 1} \\ &\cdot \left\{8 + 7\left(cleafh_3(l)\right)^6 \left(-188 + 429\left(cleafh_3(l)\right)^6\right)\right\} \end{aligned} \tag{A23}$$

The eighth derivative of the hyperbolic leaf function $cleafh_3(l)$ is as follows:

$$\begin{aligned} &\frac{d^8}{dl^8} cleafh_3(l) = 2025\left(cleafh_3(l)\right)^5 \\ &\cdot \left\{176 + 7\left(cleafh_3(l)\right)^6 \left(-152 + 143\left(cleafh_3(l)\right)^6\right)\right\} \end{aligned} \tag{A24}$$

The ninth derivative of the hyperbolic leaf function $cleafh_3(l)$ is as follows:

$$\begin{aligned} &\frac{d^9}{dl^9} cleafh_3(l) = 22275\left(cleafh_3(l)\right)^4 \sqrt{\left(cleafh_3(l)\right)^6 - 1} \\ &\cdot \left\{80 + 7\left(cleafh_3(l)\right)^6 \left(-152 + 221\left(cleafh_3(l)\right)^6\right)\right\} \end{aligned} \tag{A25}$$

The tenth derivative of the hyperbolic leaf function $cleafh_3(l)$ is as follows:

$$\begin{aligned} &\frac{d^{10}}{dl^{10}} cleafh_3(l) = 22275\left(cleafh_3(l)\right)^3 \\ &\left(-320 + 7\left(cleafh_3(l)\right)^6 \left(1600 - 5512\left(cleafh_3(l)\right)^6 + 4199\left(cleafh_3(l)\right)^{12}\right)\right) \end{aligned} \tag{A26}$$

The eleventh derivative of the hyperbolic leaf function $cleafh_3(l)$ is as follows:

$$\begin{aligned} &\frac{d^{11}}{dl^{11}} cleafh_3(l) = 66825\left(cleafh_3(l)\right)^2 \sqrt{\left(cleafh_3(l)\right)^6 - 1} \\ &\times \left(-320 + 7\left(cleafh_3(l)\right)^6\right)\left(4800 - 27560\left(cleafh_3(l)\right)^6 + 29393\left(cleafh_3(l)\right)^{12}\right) \end{aligned} \tag{A27}$$

The twelfth derivative of the hyperbolic leaf function $cleafh_3(l)$ is as follows:

$$\begin{aligned} &\frac{d^{12}}{dl^{12}} cleafh_3(l) \\ &= 42768000\ cleafh_3(l) - 1806948000\,0\ \left(cleafh_3(l)\right)^7 \\ &+ 2051848260\,00\ \left(cleafh_3(l)\right)^{13} - 4941481545\,00\ \left(cleafh_3(l)\right)^{19} \\ &+ 3162341432\,25\ \left(cleafh_3(l)\right)^{25} \end{aligned} \tag{A28}$$

The thirteenth derivative of the hyperbolic leaf function $cleafh_3(l)$ is as follows:

$$\begin{aligned} &\frac{d^{13}}{dl^{13}} cleafh_3(l) = 334125\sqrt{\left(cleafh_3(l)\right)^6 - 1} \times \\ &\left\{128 - 378560\left(cleafh_3(l)\right)^6 + 7983248\left(cleafh_3(l)\right)^{12}\right. \\ &\left. - 28099708\left(cleafh_3(l)\right)^{18} + 23661365\left(cleafh_3(l)\right)^{24}\right\} \end{aligned} \tag{A29}$$

Using the derivatives from Eqs. (A17)–(A29), the Maclaurin series of the hyperbolic leaf function $cleafh_3(l)$ is formulated as follows:

$$cleafh_3(l) = 1 + \frac{3}{2!}l^2 + \frac{45}{4!}l^4 + \frac{2295}{6!}l^6 + \frac{228825}{8!}l^8 + O(l^{10})$$
$$= 1 + \frac{3}{2}l^2 + \frac{15}{8}l^4 + \frac{51}{16}l^6 + \frac{5085}{896}l^8 + O(l^{10}) \tag{A30}$$

Using the above equation, the second derivative with respect to variable l is obtained as follows:

$$\frac{d^2}{dl^2}cleafh_3(l) = 3 + \frac{45}{2}l^2 + \frac{765}{8}l^4 + \frac{5085}{16}l^6 + O(l^8) \tag{A31}$$

Using Eq. (A30), the following equation is obtained:

$$3 \cdot (cleafh_3(l))^5 = 3 \cdot \left(1 + \frac{3}{2}l^2 + \frac{15}{8}l^4 + \frac{51}{16}l^6 + \frac{5085}{896}l^8 + O(l^{10})\right)^5$$
$$= 3 + \frac{45}{2}l^2 + \frac{765}{8}l^4 + \frac{5085}{16}l^6 + O(l^8) \tag{A32}$$

Eq. (A31) is equal to Eq. (A32). Therefore, the hyperbolic leaf function *cleafh₃(l)* satisfies Eq. (1).

Subsequently, in the case of the basis $n = 4$, the Maclaurin series is applied to the hyperbolic leaf function $cleafh_4(l)$. The first derivative of the hyperbolic leaf function $cleafh_4(l)$ is as follows:

$$\frac{d}{dl}cleafh_4(l) = \sqrt{(cleafh_4(l))^8 - 1} \tag{A33}$$

The second derivative of the hyperbolic leaf function $cleafh_4(l)$ is as follows:

$$\frac{d^2}{dl^2}cleafh_4(l) = 4 \cdot (cleafh_4(l))^7 \tag{A34}$$

The third derivative of the hyperbolic leaf function $cleafh_4(l)$ is as follows:

$$\frac{d^3}{dl^3}cleafh_4(l) = 28 \cdot (cleafh_4(l))^6 \cdot \sqrt{(cleafh_4(l))^8 - 1} \tag{A35}$$

The fourth derivative of the hyperbolic leaf function $cleafh_4(l)$ is as follows:

$$\frac{d^4}{dl^4}cleafh_4(l) = 56 \cdot (cleafh_4(l))^5 \cdot \left(5(cleafh_4(l))^8 - 3\right) \tag{A36}$$

The fifth derivative of the hyperbolic leaf function $cleafh_4(l)$ is as follows:

$$\frac{d^5}{dl^5}cleafh_4(l) = 280(cleafh_4(l))^4\left(13(cleafh_4(l))^8 - 3\right)\sqrt{(cleafh_4(l))^8 - 1} \tag{A37}$$

The sixth derivative of the hyperbolic leaf function $cleafh_4(l)$ is as follows:

$$\frac{d^6}{dl^6}cleafh_4(l) = 1120(cleafh_4(l))^3$$
$$\left(3 - 45(cleafh_4(l))^8 + 52(cleafh_4(l))^{16}\right) \tag{A38}$$

The seventh derivative of the hyperbolic leaf function $cleafh_4(l)$ is as follows:

$$\frac{d^7}{dl^7}cleafh_4(l) = 1120(cleafh_4(l))^2\sqrt{(cleafh_4(l))^8 - 1}$$
$$\cdot\left(9 - 495(cleafh_4(l))^8 + 988(cleafh_4(l))^{16}\right) \tag{A39}$$

The eighth derivative of the hyperbolic leaf function $cleafh_4(l)$ is as follows:

$$\frac{d^8}{dl^8}cleafh_4(l) = 2240\,cleafh_4(l)$$
$$\cdot\left(-9 + 2502(cleafh_4(l))^8 - 12357(cleafh_4(l))^{16} + 10868(cleafh_4(l))^{24}\right) \tag{A40}$$

The ninth derivative of the hyperbolic leaf function $cleafh_4(l)$ is as follows:

$$\frac{d^9}{dl^9}cleafh_4(l) = 2240\sqrt{(cleafh_4(l))^8 - 1}\,\cdot$$
$$\left(-9 + 22518(cleafh_4(l))^8 - 210069(cleafh_4(l))^{16} + 271700(cleafh_4(l))^{24}\right) \tag{A41}$$

The tenth derivative of the hyperbolic leaf function $cleafh_4(l)$ is as follows:

$$\frac{d^{10}}{dl^{10}}cleafh_4(l) = 313600(cleafh_4(l))^7\,\cdot$$
$$\left(-1287 + 25938(cleafh_4(l))^8 - 76587(cleafh_4(l))^{16} + 54340(cleafh_4(l))^{24}\right) \tag{A42}$$

The eleventh derivative of the hyperbolic leaf function $cleafh_4(l)$ is as follows:

$$\frac{d^{11}}{dl^{11}} cleafh_4(l) = 313600\left(cleafh_4(l)\right)^6 \sqrt{\left(cleafh_4(l)\right)^8 - 1} \cdot \left(-9009 + 389070\left(cleafh_4(l)\right)^8 - 1761501\left(cleafh_4(l)\right)^{16} + 1684540\left(cleafh_4(l)\right)^{24}\right) \quad \text{(A43)}$$

The twelfth derivative of the hyperbolic leaf function: *cleafh₄(l)* is as follows:

$$\frac{d^{12}}{dl^{12}} cleafh_4(l) = 627200\left(cleafh_4(l)\right)^5 \cdot \left(27027 + 17\left(cleafh_4(l)\right)^8\left(-162855 + 13\left(cleafh_4(l)\right)^8\left(103521 - 217953\left(cleafh_4(l)\right)^8 + 129580\left(cleafh_4(l)\right)^{16}\right)\right)\right) \quad \text{(A44)}$$

The thirteenth derivative of the hyperbolic leaf function $cleafh_4(l)$ is as follows:

$$\frac{d^{13}}{dl^{13}} cleafh_4(l) = 8153600\left(cleafh_4(l)\right)^4 \sqrt{\left(cleafh_4(l)\right)^8 - 1}\left(10395 + 17\left(cleafh_4(l)\right)^8 \cdot \left(-162855 + 2173941\left(cleafh_4(l)\right)^8 - 6320637\left(cleafh_4(l)\right)^{16} + 4794460\left(cleafh_4(l)\right)^{24}\right)\right) \quad \text{(A45)}$$

Using the derivatives from Eqs. (A33)–(A45), the Maclaurin series of the hyperbolic leaf function $cleafh_4(l)$ is formulated as follows:

$$cleafh_4(l) = 1 + 2l^2 + \frac{14}{3}l^4 + \frac{140}{9}l^6 + \frac{502}{9}l^8 + O\left(l^{10}\right) \quad \text{(A46)}$$

Using the above equation, the second derivative with respect to variable l is obtained as follows:

$$\frac{d^2}{dl^2} cleafh_4(l) = 4 + 56l^2 + \frac{1400}{3}l^4 + \frac{28112}{9}l^6 + O\left(l^8\right) \quad \text{(A47)}$$

Using Eq. (A46), the following equation is obtained:

$$4 \cdot \left(cleafh_4(l)\right)^7 = 4 \cdot \left(1 + 2l^2 + \frac{14}{3}l^4 + \frac{140}{9}l^6 + \frac{502}{9}l^8 + O\left(l^{10}\right)\right)^7 = 4 + 56l^2 + \frac{1400}{3}l^4 + \frac{28112}{9}l^6 + O\left(l^8\right) \quad \text{(A48)}$$

Eq. (A47) is equal to Eq. (A48). Therefore, the hyperbolic leaf function $cleafh_4(l)$ satisfies Eq. (1).

Subsequently, in the case of the basis $n = 5$, the Maclaurin series is applied to the hyperbolic leaf function $cleafh_5(l)$. The first derivative of the hyperbolic leaf function $cleafh_5(l)$ is as follows:

$$\frac{d}{dl} cleaf_5(l) = \sqrt{\left(cleaf_5(l)\right)^{10} - 1} \quad \text{(A49)}$$

The second derivative of the hyperbolic leaf function $cleafh_5(l)$ is as follows:

$$\frac{d^2}{dl^2} cleafh_5(l) = 5 \cdot \left(cleafh_5(l)\right)^9 \quad \text{(A50)}$$

The third derivative of the hyperbolic leaf function $cleafh_5(l)$ is as follows:

$$\frac{d^3}{dl^3} cleafh_5(l) = 45 \cdot \left(cleafh_5(l)\right)^8 \cdot \sqrt{\left(cleafh_5(l)\right)^{10} - 1} \quad \text{(A51)}$$

The fourth derivative of the hyperbolic leaf function $cleafh_5(l)$ is as follows:

$$\frac{d^4}{dl^4} cleafh_5(l) = 45 \cdot \left(cleafh_5(l)\right)^7 \cdot \left(-8 + 13\left(cleafh_5(l)\right)^{10}\right) \quad \text{(A52)}$$

The fifth derivative of the hyperbolic leaf function $cleafh_5(l)$ is as follows:

$$\frac{d^5}{dl^5} cleafh_5(l) = 45\left(cleafh_5(l)\right)^6 \left(-56 + 221\left(cleafh_5(l)\right)^{10}\right)\sqrt{\left(cleafh_5(l)\right)^{10} - 1} \quad \text{(A53)}$$

The sixth derivative of the hyperbolic leaf function $cleafh_5(l)$ is as follows:

$$\frac{d^6}{dl^6} cleafh_5(l) = 135\left(cleafh_5(l)\right)^5 \cdot \left(112 - 1384\left(cleafh_5(l)\right)^{10} + 1547\left(cleafh_5(l)\right)^{20}\right) \quad \text{(A54)}$$

The seventh derivative of the hyperbolic leaf function $cleafh_5(l)$ is as follows:

$$\frac{d^7}{dl^7} cleafh_5(l) = 675\left(cleafh_5(l)\right)^4 \sqrt{\left(cleafh_5(l)\right)^{10} - 1} \cdot \left(112 - 4152\left(cleafh_5(l)\right)^{10} + 7735\left(cleafh_5(l)\right)^{20}\right) \quad \text{(A55)}$$

The eighth derivative of the hyperbolic leaf function $cleafh_5(l)$ is as follows:

$$\frac{d^8}{dl^8}cleafh_5(l) = 675\left(cleafh_5(l)\right)^3 \cdot\left(-448 + 59136\left(cleafh_5(l)\right)^{10} - 264528\left(cleafh_5(l)\right)^{20} + 224315\left(cleafh_5(l)\right)^{30}\right) \tag{A56}$$

The ninth derivative of the hyperbolic leaf function *cleafh₅(l)* is as follows:

$$\frac{d^9}{dl^9}cleafh_5(l) = 2025\left(cleafh_5(l)\right)^2\sqrt{\left(cleafh_5(l)\right)^{10} - 1} \cdot\left(-448 + 256256\left(cleafh_5(l)\right)^{10} - 2028048\left(cleafh_5(l)\right)^{20} + 2467465\left(cleafh_5(l)\right)^{30}\right) \tag{A57}$$

The tenth derivative of the hyperbolic leaf function $cleafh_5(l)$ is as follows:

$$\frac{d^{10}}{dl^{10}}cleafh_5(l) = 2025\,cleafh_5(l)\cdot\left(896 - 3078208\left(cleafh_5(l)\right)^{10} + 48973408\left(cleafh_5(l)\right)^{20} - 133716176\left(cleafh_5(l)\right)^{30} + 91296205\left(cleafh_5(l)\right)^{40}\right) \tag{A58}$$

The eleventh derivative of the hyperbolic leaf function $cleafh_5(l)$ is as follows:

$$\frac{d^{11}}{dl^{11}}cleafh_5(l) = 2025\sqrt{\left(cleafh_5(l)\right)^{10} - 1}\cdot\left(896 + 11\left(cleafh_5(l)\right)^{10} \left(-3078208 + 93494688\left(cleafh_5(l)\right)^{10} - 376836496\left(cleafh_5(l)\right)^{20} + 340285855\left(cleafh_5(l)\right)^{30}\right)\right) \tag{A59}$$

Using the derivatives from Eqs. (A49)–(A59), the Maclaurin series of the hyperbolic leaf function $cleafh_5(l)$ is formulated as follows:

$$cleafh_5(l) = 1 + \frac{5}{2}l^2 + \frac{75}{8}l^4 + \frac{825}{16}l^6 + \frac{277125}{896}l^8 + O\left(l^{10}\right) \tag{A60}$$

Using the above equation, the second derivative with respect to variable l is obtained as follows:

$$\frac{d^2}{dl^2}cleafh_5(l) = 5 + \frac{225}{2}l^2 + \frac{12375}{8}l^4 + \frac{277125}{16}l^6 + O\left(l^8\right) \tag{A61}$$

Using Eq. (A60), the following equation is obtained:

$$\begin{aligned} 5\cdot\left(cleafh_5(l)\right)^9 &= 5\cdot\left(1 + \frac{5}{2}l^2 + \frac{75}{8}l^4 + \frac{825}{16}l^6 + \frac{277125}{896}l^8 + O\left(l^{10}\right)\right)^9 \\ &= 5 + \frac{225}{2}l^2 + \frac{12375}{8}l^4 + \frac{277125}{16}l^6 + O\left(l^8\right) \end{aligned} \tag{A62}$$

Eq. (A61) is equal to Eq. (A62). Therefore, the hyperbolic leaf function $cleafh_5(l)$ satisfies Eq. (1).

Appendix B

In this section, the relation between the leaf function $cleaf_2(l)$ and the hyperbolic leaf function $cleafh_2(l)$ is described. The following polynomial is considered:

$$xy = 1 \tag{B1}$$

The following equation is obtained by differentiating the above equation with respect to variable x:

$$\frac{dy}{dx} = -\frac{1}{x^2} \tag{B2}$$

Using Eqs. (B1) - (B2), the following equation is obtained:

$$\begin{aligned} \frac{1}{\sqrt{1-y^4}}\frac{dy}{dx} &= \frac{1}{\sqrt{1-\left(\frac{1}{x}\right)^4}}\left(-\frac{1}{x^2}\right) \\ &= -\frac{x^2}{\sqrt{x^4-1}}\frac{1}{x^2} = -\frac{1}{\sqrt{x^4-1}} \end{aligned} \tag{B3}$$

The following equation is obtained from the above equation:

$$\frac{dy}{\sqrt{1-y^4}} + \frac{dx}{\sqrt{x^4-1}} = 0 \tag{B4}$$

Variables x and y are defined by the following equations:

$$x = cleafh_2(l) \tag{B5}$$

$$y = cleaf_2(l) \tag{B6}$$

The domain of variable l is as follows:

$$4m\eta_2 \le l \le (4m+2)\eta_2 \tag{B7}$$

The number m represent the integer. The following equation is obtained by differentiating the above equation with respect to variable l:

$$\frac{dx}{dl}=\sqrt{(cleafh_2(l))^4-1}=\sqrt{x^4-1} \tag{B8}$$

$$\frac{dy}{dl}=\sqrt{1-(cleaf_2(l))^4}=-\sqrt{1-y^4} \tag{B9}$$

The following equation is obtained by substituting Eqs. (B8)–(B9) into Eq. (B4):

$$\begin{aligned}\frac{dy}{\sqrt{1-y^4}}+\frac{dx}{\sqrt{x^4-1}}&=\frac{1}{\sqrt{1-y^4}}\frac{dy}{dl}dl+\frac{1}{\sqrt{x^4-1}}\frac{dx}{dl}dl\\&=-\frac{1}{\sqrt{1-y^4}}\sqrt{1-y^4}dl+\frac{1}{\sqrt{x^4-1}}\sqrt{x^4-1}dl=0\end{aligned} \tag{B10}$$

On the other hand, the domain of variable l is as follows:

$$(4m-2)\eta_2\le l\le 4m\eta_2 \tag{B11}$$

The following equation is obtained by differentiating the above equation with respect to variable l:

$$\frac{dx}{dl}=\sqrt{(cleafh_2(l))^4-1}=-\sqrt{x^4-1} \tag{B12}$$

$$\frac{dy}{dl}=\sqrt{1-(cleaf_2(l))^4}=\sqrt{1-y^4} \tag{B13}$$

The following equation is obtained by substituting Eqs. (B12)–(B13) into Eq. (B4):

$$\begin{aligned}\frac{dy}{\sqrt{1-y^4}}+\frac{dx}{\sqrt{x^4-1}}&=\frac{1}{\sqrt{1-y^4}}\frac{dy}{dl}dl+\frac{1}{\sqrt{x^4-1}}\frac{dx}{dl}dl\\&=\frac{1}{\sqrt{1-y^4}}\sqrt{1-y^4}dl-\frac{1}{\sqrt{x^4-1}}\sqrt{x^4-1}dl=0\end{aligned} \tag{B14}$$

Eqs. (B5) and (B6) satisfy Eq. (B1). Therefore, the following relation is obtained:

$$cleaf_2(l)\cdot cleafh_2(l)=1 \tag{B15}$$

Appendix C

In this section, the relation between the hyperbolic leaf function $sleafh_2(l)$ and the hyperbolic leaf function $cleafh_2(l)$ is described. The following polynomial is considered:

$$-yx^2+y-x^2=1 \tag{C1}$$

The following equation is obtained from the above equation:

$$y=\frac{1+x^2}{1-x^2} \tag{C2}$$

The following equation is obtained by differentiating the above equation with respect to variable x:

$$\frac{dy}{dx}=\frac{4x}{(1-x^2)^2} \tag{C3}$$

Using Eqs. (C2)–(C3), the following equation is obtained:

$$\begin{aligned}\frac{1}{\sqrt{y^4-1}}\frac{dy}{dx}&=\frac{1}{\sqrt{\left(\frac{1+x^2}{1-x^2}\right)^4-1}}\frac{4x}{(1-x^2)^2}\\&=\frac{(1-x^2)^2}{2\sqrt{2}|x|\sqrt{1+x^4}}\frac{4x}{(1-x^2)^2}=\frac{\sqrt{2}}{\sqrt{1+x^4}}\frac{x}{|x|}\end{aligned} \tag{C4}$$

where the above equation is applied to $\sqrt{x^2}=|x|$. In the inequality $x \geqq 0$, the above equation is transformed as follows:

$$\frac{dy}{\sqrt{y^4-1}}-\sqrt{2}\frac{dx}{\sqrt{1+x^4}}=0 \tag{C5}$$

The variables x and y are defined by the following equations:

$$x=sleafh_2(l) \tag{C6}$$

$$y=cleafh_2(\sqrt{2}l) \tag{C7}$$

In the condition $x=sleafh_2(l)\ge 0$, the domain of variable l is as follows:

$$4m\eta_2 \le l \le (4m+2)\eta_2 \tag{C8}$$

The number m represent the integer. The following equation is obtained by differentiating the above equation with respect to variable l:

$$\frac{dx}{dl} = \sqrt{1+(sleafh_2(l))^4} = \sqrt{1+x^4} \tag{C9}$$

$$\frac{dy}{dl} = \sqrt{2}\sqrt{\left(cleafh_2\left(\sqrt{2}l\right)\right)^4 - 1} = \sqrt{2}\sqrt{y^4-1} \tag{C10}$$

The following equation is obtained by substituting Eqs. (C8)–(C9) into Eq. (C5):

$$\frac{dy}{\sqrt{y^4-1}} - \sqrt{2}\frac{dx}{\sqrt{x^4+1}} = \frac{1}{\sqrt{y^4-1}}\frac{dy}{dl}dl - \sqrt{2}\frac{1}{\sqrt{x^4+1}}\frac{dx}{dl}dl$$
$$= \frac{1}{\sqrt{y^4-1}}\sqrt{2}\sqrt{y^4-1}dl - \frac{\sqrt{2}}{\sqrt{x^4+1}}\sqrt{x^4+1}dl = 0 \tag{C11}$$

In the inequality $x < 0$, the above equation is transformed as follows:

$$\frac{dy}{\sqrt{y^4-1}} + \sqrt{2}\frac{dx}{\sqrt{1+x^4}} = 0 \tag{C12}$$

In the condition $x = sleafh_2(l) < 0$, the domain of variable l is as follows:

$$(4m-2)\eta_2 \le l \le 4m\eta_2 \tag{C13}$$

Using Eqs. (C6) and (C7), the following equation is obtained by differentiating the above equation with respect to variable l.

$$\frac{dx}{dl} = \sqrt{1+(sleafh_2(l))^4} = \sqrt{1+x^4} \tag{C14}$$

$$\frac{dy}{dl} = \sqrt{2}\sqrt{\left(cleafh_2\left(\sqrt{2}l\right)\right)^4 - 1} = -\sqrt{2}\sqrt{y^4-1} \tag{C15}$$

The following equation is obtained by substituting Eqs. (C14)–(C15) into Eq. (C12).

$$\frac{dy}{\sqrt{y^4-1}} + \sqrt{2}\frac{dx}{\sqrt{x^4+1}} = \frac{1}{\sqrt{y^4-1}}\frac{dy}{dl}dl + \sqrt{2}\frac{1}{\sqrt{x^4+1}}\frac{dx}{dl}dl$$
$$= \frac{1}{\sqrt{y^4-1}}\left(-\sqrt{2}\sqrt{y^4-1}\right)dl + \frac{\sqrt{2}}{\sqrt{x^4+1}}\sqrt{x^4+1}dl = 0 \tag{C16}$$

Eqs. (C6) and (C7) satisfy Eq. (C1). Therefore, the following relation is obtained:

$$cleafh_2\left(\sqrt{2}l\right) = \frac{1+(sleafh_2(l))^2}{1-(sleafh_2(l))^2} \tag{C17}$$

Appendix D

In this section, the relation between the hyperbolic leaf function: $sleafh_3(l)$ and the hyperbolic leaf function: $cleafh_3(l)$ is described. The following polynomial is considered:

$$x^2 - y^2 - 2x^2y^2 = 1 \tag{D1}$$

The above equation is solved for variable y.

$$y = \pm\frac{\sqrt{x^2-1}}{\sqrt{2x^2+1}} \tag{D2}$$

The following equation is obtained by differentiating the above equation with respect to variable x:

$$\frac{dy}{dx} = \pm\frac{3x}{\sqrt{x^2-1}\left(1+2x^2\right)^{\frac{3}{2}}} \tag{D3}$$

Using Eqs. (D2)–(D3), the following equation is obtained:

$$\frac{1}{\sqrt{1+y^6}}\frac{dy}{dx} = \pm\frac{1}{\sqrt{1+\left(\frac{\sqrt{x^2-1}}{\sqrt{2x^2+1}}\right)^6}}\frac{3x}{\sqrt{x^2-1}\left(1+2x^2\right)^{\frac{3}{2}}}$$
$$= \pm\frac{\left(1+2x^2\right)^{\frac{3}{2}}}{3\sqrt{x^2+x^4+x^6}}\frac{3x}{\sqrt{x^2-1}\left(1+2x^2\right)^{\frac{3}{2}}}$$
$$= \pm\frac{x}{\sqrt{x^2-1}\sqrt{x^2+x^4+x^6}} = \pm\frac{x}{|x|\sqrt{x^6-1}} = \pm\frac{1}{\sqrt{x^6-1}} \tag{D4}$$

The following equation is obtained from the above equation:

$$\frac{dy}{\sqrt{1+y^6}} \pm \frac{dx}{\sqrt{x^6-1}} = 0 \tag{D5}$$

Variables x and y are defined by the following equations:

$$x = \mathit{cleafh}_3(l) \tag{D6}$$

$$y = \mathit{sleafh}_3(l) \tag{D7}$$

The following equation is obtained by differentiating the above equation with respect to variable l:

$$\frac{dx}{dl} = \pm\sqrt{(\mathit{cleafh}_3(l))^6 - 1} = \pm\sqrt{x^6-1} \tag{D8}$$

$$\frac{dy}{dl} = \sqrt{1+(\mathit{sleafh}_3(l))^6} = \sqrt{1+y^6} \tag{D9}$$

Using Eq. (D5), (D8) and (D9), the following relation is obtained:

$$\begin{gathered}(\mathit{cleafh}_3(l))^2 - (\mathit{sleafh}_3(l))^2 - 2(\mathit{cleafh}_3(l))^2(\mathit{sleafh}_3(l))^2 = 1 \\ (4m-1)\eta_3 < l < (4m+1)\eta_3\end{gathered} \tag{D10}$$

Appendix E

To prove the addition theorem of Eq. (64), we define the following equation:

$$l_1 + l_2 = c \tag{E1}$$

Symbol c represents the arbitrary constant. Using Eqs. (E1) and (64), the following equation is obtained:

$$\begin{aligned}&\mathit{cleafh}_2(c) = \\ &\frac{2\mathit{cleafh}_2(l_1)\mathit{cleafh}_2(c-l_1) + \sqrt{(\mathit{cleafh}_2(l_1))^4-1}\sqrt{(\mathit{cleafh}_2(c-l_1))^4-1}}{1+(\mathit{cleafh}_2(l_1))^2+(\mathit{cleafh}_2(c-l_1))^2-(\mathit{cleafh}_2(l_1))^2(\mathit{cleafh}_2(c-l_1))^2}\end{aligned} \tag{E2}$$

The right side of the above equation is defined as follows:

$$\begin{aligned}&F(l_1) = \\ &\frac{2\mathit{cleafh}_2(l_1)\mathit{cleafh}_2(c-l_1) + \sqrt{(\mathit{cleafh}_2(l_1))^4-1}\sqrt{(\mathit{cleafh}_2(c-l_1))^4-1}}{1+(\mathit{cleafh}_2(l_1))^2+(\mathit{cleafh}_2(c-l_1))^2-(\mathit{cleafh}_2(l_1))^2(\mathit{cleafh}_2(c-l_1))^2}\end{aligned} \tag{E3}$$

The symbol *cleafh*$_2$*(c)* is just a constant. The following equation is derived from Eq. (E2) and Eq. (E3):

$$F(l_1) = \mathit{cleafh}_2(c) \tag{E4}$$

Therefore, function $F(l_1)$ also has to be a constant.

$$\frac{\partial F(l_1)}{\partial l_1} = 0 \tag{E5}$$

If the above equation is satisfied, function $F(l_1)$ becomes a constant. To prove Eq. (E5), function $F(l_1)$ is differentiated with respect to variable l_1.

$$\begin{aligned}\frac{\partial F(l_1)}{\partial l_1} &= \frac{\left\{2\mathit{cleafh}_2(l_1)\mathit{cleafh}_2(c-l_1) + \sqrt{(\mathit{cleafh}_2(l_1))^4-1}\sqrt{(\mathit{cleafh}_2(c-l_1))^4-1}\right\}'}{\left\{1+(\mathit{cleafh}_2(l_1))^2+(\mathit{cleafh}_2(c-l_1))^2-(\mathit{cleafh}_2(l_1))^2(\mathit{cleafh}_2(c-l_1))^2\right\}^2} \\ &\times\left\{1+(\mathit{cleafh}_2(l_1))^2+(\mathit{cleafh}_2(c-l_1))^2-(\mathit{cleafh}_2(l_1))^2(\mathit{cleafh}_2(c-l_1))^2\right\} \\ &+ \frac{\left\{2\mathit{cleafh}_2(l_1)\mathit{cleafh}_2(c-l_1) + \sqrt{(\mathit{cleafh}_2(l_1))^4-1}\sqrt{(\mathit{cleafh}_2(c-l_1))^4-1}\right\}}{\left\{1+(\mathit{cleafh}_2(l_1))^2+(\mathit{cleafh}_2(c-l_1))^2-(\mathit{cleafh}_2(l_1))^2(\mathit{cleafh}_2(c-l_1))^2\right\}^2} \\ &\times\left\{1+(\mathit{cleafh}_2(l_1))^2+(\mathit{cleafh}_2(c-l_1))^2-(\mathit{cleafh}_2(l_1))^2(\mathit{cleafh}_2(c-l_1))^2\right\}'\end{aligned} \tag{E6}$$

On the other hand, the following equation is obtained:

$$\begin{aligned}&\left\{2\mathit{cleafh}_2(l_1)\mathit{cleafh}_2(c-l_1) + \sqrt{(\mathit{cleafh}_2(l_1))^4-1}\sqrt{(\mathit{cleafh}_2(c-l_1))^4-1}\right\}' \\ &= 2\mathit{cleafh}_2(l_1)\left\{(\mathit{cleafh}_2(l_1))^2-1\right\}\sqrt{(\mathit{cleafh}_2(c-l_1))^4-1} \\ &- 2\mathit{cleafh}_2(c-l_1)\left\{(\mathit{cleafh}_2(c-l_1))^2-1\right\}\sqrt{(\mathit{cleafh}_2(l_1))^4-1}\end{aligned} \tag{E7}$$

$$\begin{aligned}&\left\{1+(\mathit{cleafh}_2(l_1))^2+(\mathit{cleafh}_2(c-l_1))^2-(\mathit{cleafh}_2(l_1))^2(\mathit{cleafh}_2(c-l_1))^2\right\}' \\ &= 2\mathit{cleafh}_2(l_1)\mathit{cleafh}_2(c-l_1)\left\{\mathit{cleafh}_2(l_1)\sqrt{(\mathit{cleafh}_2(c-l_1))^4-1} - \mathit{cleafh}_2(c-l_1)\sqrt{(\mathit{cleafh}_2(l_1))^4-1}\right\} \\ &+ 2\mathit{cleafh}_2(l_1)\sqrt{(\mathit{cleafh}_2(l_1))^4-1} - 2\mathit{cleafh}_2(c-l_1)\sqrt{(\mathit{cleafh}_2(c-l_1))^4-1}\end{aligned} \tag{E8}$$

By substituting Eqs. (E7) and (E8) into Eq. (E6), Eq. (E5) is obtained. Function $F(l_1)$ does not depend on variable l_1. Therefore, the following equation is obtained:

$$F(l_1) = F(0) \tag{E9}$$

By substituting $l_1 = 0$ into Eq. (E3), the following equation is obtained:

$$F(0) = \frac{2cleafh_2(0)cleafh_2(c) + \sqrt{(cleafh_2(0))^4 - 1}\sqrt{(cleafh_2(c))^4 - 1}}{1 + (cleafh_2(0))^2 + (cleafh_2(c))^2 - (cleafh_2(0))^2(cleafh_2(c))^2}$$
$$= \frac{2cleafh_2(c) + \sqrt{1-1}\sqrt{(cleafh_2(c))^4 - 1}}{1 + 1 + (cleafh_2(c))^2 - (cleafh_2(c))^2} = cleafh_2(c) \tag{E10}$$

From Eqs. (E9) and (E10), Eq. (E4) is obtained. The proof is the same as Eqs. (67), (70), and (73).

Appendix F

Using imaginary number i, Eqs. (54)–(57) can be derived by using Eqs. (65)–(67) in Ref. [2]. As shown in Eq. (84), the hyperbolic leaf function is related to the leaf function through imaginary number i. By replacing variable l into variable $i \cdot l$, the following equation is obtained:

$$cleafh_n(-l) = cleaf_n(i \cdot l) \tag{F1}$$

The hyperbolic leaf function $cleafh_n(l)$ is the even function. The following equation is obtained:

$$cleafh_n(l) = cleaf_n(i \cdot l) \tag{F2}$$

In a similar manner, as described in the above procedure, the following equation is obtained by using Eqs. (30)–(32) in Ref. [3]:

$$sleaf_{2m-1}(-l) = i \cdot sleafh_{2m-1}(i \cdot l) \quad (m = 1,2,3,,\cdots) \tag{F3}$$

$$sleaf_{2m}(-l) = i \cdot sleaf_{2m}(i \cdot l) \quad (m = 1,2,3,\cdots) \tag{F4}$$

$$sleafh_{2m}(-l) = i \cdot sleafh_{2m}(i \cdot l) \quad (m = 1,2,3,\cdots) \tag{F5}$$

The hyperbolic leaf function $sleafh_n(l)$ is the odd function. The following equation is obtained:

$$i \cdot sleaf_{2m-1}(l) = sleafh_{2m-1}(i \cdot l) \quad (m = 1,2,3,,\cdots) \tag{F6}$$

$$i \cdot sleaf_{2m}(l) = sleaf_{2m}(i \cdot l) \quad (m = 1,2,3,\cdots) \tag{F7}$$

$$i \cdot sleafh_{2m}(l) = sleafh_{2m}(i \cdot l) \quad (m = 1,2,3,\cdots) \tag{F8}$$

In the case of the basis $n = 1$, the following equation between the leaf function and the hyperbolic leaf function is obtained:

$$(sleaf(i \cdot l))^2 + (cleaf(i \cdot l))^2 = 1 \tag{F9}$$

By substituting Eqs. (F6) and (F2) into Eq. (F9), the following equation is obtained:

$$(i \cdot sleafh_1(l))^2 + (cleafh_1(l))^2 = 1 \tag{F10}$$

$$(cleafh_1(l))^2 - (sleafh_1(l))^2 = 1 \tag{F11}$$

The above equation has the same relation between the hyperbolic function *sinh(l)* and the hyperbolic function *cosh(l)*. In the case of the basis $n = 2$, the leaf function: $sleaf_2(l)$ is related to the leaf function: $cleaf_2(l)$.

$$(sleaf_2(l))^2 + (cleaf_2(l))^2 + (sleaf_2(l))^2 \cdot (cleaf_2(l))^2 = 1 \tag{F12}$$

By replacing variable l into variable $i \cdot l$, the following equation is obtained:

$$(sleaf_2(i \cdot l))^2 + (cleaf_2(i \cdot l))^2 + (sleaf_2(i \cdot l))^2 \cdot (cleaf_2(i \cdot l))^2 = 1 \tag{F13}$$

By substituting Eqs. (F7) and (F2) into the above equation, the following equation is obtained:

$$(i \cdot sleaf_2(l))^2 + (cleafh_2(l))^2 + (i \cdot sleaf_2(l))^2 \cdot (cleafh_2(l))^2 = 1 \tag{F14}$$

$$-(sleaf_2(l))^2 + (cleafh_2(l))^2 - (sleaf_2(l))^2 \cdot (cleafh_2(l))^2 = 1 \tag{F15}$$

By substituting variable l into variable $\sqrt{2}l$, the following equation is obtained:

$$-(sleaf_2(\sqrt{2} \cdot l))^2 + (cleafh_2(\sqrt{2} \cdot l))^2 - (sleaf_2(\sqrt{2} \cdot l))^2 \cdot (cleafh_2(\sqrt{2} \cdot l))^2 = 1 \tag{F16}$$

By substituting Eq. (33) in Ref [3], the following equation is obtained:

$$-\frac{2(sleafh_2(l))^2}{1+(sleafh_2(l))^4}+\left(cleafh_2\left(\sqrt{2}\cdot l\right)\right)^2-\frac{2(sleafh_2(l))^2}{1+(sleafh_2(l))^4}\cdot\left(cleafh_2\left(\sqrt{2}\cdot l\right)\right)^2=1 \tag{F17}$$

The above equation is simplified as follows:

$$\left\{1-(sleafh_2(l))^2\right\}^2\left(cleafh_2\left(\sqrt{2}\cdot l\right)\right)^2-\left\{1+(sleafh_2(l))^2\right\}^2=0 \tag{F18}$$

Eq. (56) is obtained from the above equation.

In the case of the basis $n = 3$, the leaf function $sleaf_3(l)$ is related to the leaf function $cleaf_3(l)$.

$$(sleaf_3(l))^2+(cleaf_3(l))^2+2\cdot(sleaf_3(l))^2\cdot(cleaf_3(l))^2=1 \tag{F19}$$

By replacing variable l into variable $i \cdot l$, the following equation is obtained:

$$(sleaf_3(i\cdot l))^2+(cleaf_3(i\cdot l))^2+2\cdot(sleaf_3(i\cdot l))^2\cdot(cleaf_3(i\cdot l))^2=1 \tag{F20}$$

By substituting Eq. (F2) and (F6) into Eq. (F20), the following equation is obtained:

$$(i\cdot sleafh_3(l))^2+(cleafh_3(l))^2+2\cdot(i\cdot sleafh_3(l))^2\cdot(cleafh_3(l))^2=1 \tag{F21}$$

$$(cleafh_3(l))^2-(sleafh_3(l))^2-2\cdot(sleafh_3(l))^2\cdot(cleafh_3(l))^2=1 \tag{F22}$$

The Eq. (57) is obtained.

Appendix G

In the case of the basis $n = 2$, the addition theorem of the leaf function $sleaf_2(l)$ is obtained as follows:

$$sleaf_2(l_1\pm l_2)=\frac{sleaf_2(l_1)\sqrt{1-(sleaf_2(l_2))^4}\pm sleaf_2(l_2)\sqrt{1-(sleaf_2(l_1))^4}}{1+(sleaf_2(l_1))^2(sleaf_2(l_2))^2} \tag{G1}$$

By replacing variable l into variable $i \cdot l$, the following equation is obtained:

$$sleaf_2(i\cdot l_1\pm i\cdot l_2)=\frac{sleaf_2(i\cdot l_1)\sqrt{1-(sleaf_2(i\cdot l_2))^4}\pm sleaf_2(i\cdot l_2)\sqrt{1-(sleaf_2(i\cdot l_1))^4}}{1+(sleaf_2(i\cdot l_1))^2(sleaf_2(i\cdot l_2))^2} \tag{G2}$$

By substituting Eqs. (F7) into the above equation, the following equation is obtained:

$$i\cdot sleaf_2(l_1\pm l_2)=\frac{i\cdot sleaf_2(l_1)\sqrt{1-(i\cdot sleaf_2(l_2))^4}\pm i\cdot sleaf_2(l_2)\sqrt{1-(i\cdot sleaf_2(l_1))^4}}{1+(i\cdot sleaf_2(l_1))^2(i\cdot sleaf_2(l_2))^2} \tag{G3}$$

The above equation is simplified as equation (G1).

In the case of the basis $n = 2$, the addition theorem of the leaf function $cleaf_2(l)$ is obtained as follows:

$$cleaf_2(l_1+l_2)=\frac{cleaf_2(l_1)\sqrt{1-(sleaf_2(l_2))^4}-sleaf_2(l_2)\sqrt{1-(cleaf_2(l_1))^4}}{1+(cleaf_2(l_1))^2(sleaf_2(l_2))^2} \tag{G4}$$

By replacing variable l into variable $i \cdot l$, the following equation is obtained:

$$cleaf_2(i\cdot l_1+i\cdot l_2)=\frac{cleaf_2(i\cdot l_1)\sqrt{1-(sleaf_2(i\cdot l_2))^4}-sleaf_2(i\cdot l_2)\sqrt{1-(cleaf_2(i\cdot l_1))^4}}{1+(cleaf_2(i\cdot l_1))^2(sleaf_2(i\cdot l_2))^2} \tag{G5}$$

By substituting Eq. (F2) and Eq. (F7) into the above equation, the following equation is obtained:

$$
\begin{aligned}
& cleafh_2(l_1+l_2) \\
&= \frac{cleafh_2(l_1)\sqrt{1-(i\cdot sleaf_2(l_2))^4} - i\cdot sleaf_2(l_2)\sqrt{1-(cleafh_2(l_1))^4}}{1+(cleafh_2(l_1))^2(i\cdot sleaf_2(l_2))^2} \\
&= \frac{cleafh_2(l_1)\sqrt{1-(sleaf_2(l_2))^4} - i\cdot sleaf_2(l_2)\sqrt{1-(cleafh_2(l_1))^4}}{1-(cleafh_2(l_1))^2(sleaf_2(l_2))^2}
\end{aligned} \tag{G6}
$$

The range of the hyperbolic leaf function is as follows:

$$
cleafh_2(l) \geq 1 \tag{G7}
$$

The root of the second term becomes negative. Therefore, Eq. (G6) is defined as follows:

$$
\begin{aligned}
& cleafh_2(l_1+l_2) \\
&= \frac{cleafh_2(l_1)\sqrt{1-(sleaf_2(l_2))^4} - i\cdot sleaf_2(l_2)\cdot i\cdot\sqrt{(cleafh_2(l_1))^4-1}}{1-(cleafh_2(l_1))^2(sleaf_2(l_2))^2} \\
&= \frac{cleafh_2(l_1)\sqrt{1-(sleaf_2(l_2))^4} + sleaf_2(l_2)\sqrt{(cleafh_2(l_1))^4-1}}{1-(cleafh_2(l_1))^2(sleaf_2(l_2))^2}
\end{aligned} \tag{G8}
$$

The following equation is obtained from Eq. (F15):

$$
sleaf_2(l) = \pm\sqrt{\frac{(cleafh_2(l))^2-1}{(cleafh_2(l))^2+1}} \tag{G9}
$$

By substituting the above equation into Eq. (G8), the following equation is obtained:

$$
\begin{aligned}
& cleafh_2(l_1+l_2) \\
&= \frac{cleafh_2(l_1)\sqrt{1-\left(\frac{(cleafh_2(l_2))^2-1}{(cleafh_2(l_2))^2+1}\right)^2} \pm \sqrt{\frac{(cleafh_2(l_2))^2-1}{(cleafh_2(l_2))^2+1}}\sqrt{(cleafh_2(l_1))^4-1}}{1-(cleafh_2(l_1))^2\left(\frac{(cleafh_2(l_2))^2-1}{(cleafh_2(l_2))^2+1}\right)}
\end{aligned} \tag{G10}
$$

By multiplying the numerator and the denominator by $(cleafh_2(l_2))^2+1$, the above equation is simplified as follows:

$$
\begin{aligned}
& cleafh_2(l_1+l_2) \\
&= \frac{cleafh_2(l_1)\sqrt{((cleafh_2(l_2))^2+1)^2-((cleafh_2(l_2))^2-1)^2} \pm \sqrt{(cleafh_2(l_1))^4-1}\sqrt{(cleafh_2(l_2))^4-1}}{(cleafh_2(l_2))^2+1-(cleafh_2(l_1))^2((cleafh_2(l_2))^2-1)} \\
&= \frac{cleafh_2(l_1)\sqrt{((cleafh_2(l_2))^2+1)^2-((cleafh_2(l_2))^2-1)^2} \pm \sqrt{(cleafh_2(l_1))^4-1}\sqrt{(cleafh_2(l_2))^4-1}}{(cleafh_2(l_2))^2+1-(cleafh_2(l_1))^2((cleafh_2(l_2))^2-1)} \\
&= \frac{cleafh_2(l_1)\sqrt{4(cleafh_2(l_2))^2} \pm \sqrt{(cleafh_2(l_1))^4-1}\sqrt{(cleafh_2(l_2))^4-1}}{(cleafh_2(l_2))^2+1-(cleafh_2(l_1))^2((cleafh_2(l_2))^2-1)} \\
&= \frac{2cleafh_2(l_1)cleafh_2(l_2) \pm \sqrt{(cleafh_2(l_1))^4-1}\sqrt{(cleafh_2(l_2))^4-1}}{1+(cleafh_2(l_1))^2+(cleafh_2(l_2))^2-(cleafh_2(l_1))^2(cleafh_2(l_2))^2} \\
&= \frac{2cleafh_2(l_1)cleafh_2(l_2) + cleafh_2'(l_1)cleafh_2'(l_2)}{1+(cleafh_2(l_1))^2+(cleafh_2(l_2))^2-(cleafh_2(l_1))^2(cleafh_2(l_2))^2}
\end{aligned} \tag{G11}
$$

In the above equation, the superscript prime ‘ of the hyperbolic leaf function represents the derivative with respect to variable *l*.

Appendix H

In this section, the relation between the hyperbolic function *cosh(l)* (=$cleafh_1(l)$) and the hyperbolic leaf function $cleafh_n(l)$ is described. The following equation is considered:

$$
(cleafh_n(l))^n = \cosh(n\theta) \quad n = 1, 2, 3, \cdots \tag{H1}
$$

Using the above equation, the following equation is obtained:

$$
\begin{aligned}
\theta &= \frac{1}{n} ar\cosh\left((cleafh_n(l))^n\right) \\
&= \frac{1}{n}\ln\left((cleafh_n(l))^n + \sqrt{(cleafh_n(l))^{2n}-1}\right) \\
& n = 1, 2, 3, \cdots
\end{aligned} \tag{H2}
$$

The above equation is differentiated with respect to variable *l*.

$$
\begin{aligned}
\frac{d\theta}{dl} &= \frac{n(cleafh_n(l))^{n-1}}{n\sqrt{(cleafh_n(l))^{2n}-1}}\sqrt{(cleafh_n(l))^{2n}-1} \\
&= (cleafh_n(l))^{n-1}
\end{aligned} \tag{H3}
$$

The following equation is obtained by integrating the above equation from *0* to *l*:

$$
\theta = \int_0^l (cleafh_n(t))^{n-1} dt \tag{H4}
$$

Using Eqs. (H1) and (H4), the following equation is obtained:

$$\left(cleafh_n(l)\right)^n = \cosh\left(n\int_0^l \left(cleafh_n(t)\right)^{n-1} dt\right) \tag{H5}$$
$$n = 1, 2, 3, \cdots$$

Note that the above equation is satisfied with the inequality: *cleafh*$_n$*(l)* ≧ *1*, if the basis *n* is odd number.

Appendix I

The integration of the hyperbolic leaf function: $(cleafh_n(l))^{n-1}$ is obtained as follows:

$$\begin{aligned}
&\frac{n}{2}\int_0^l \left(cleafh_n(t)\right)^{n-1} dt \\
&= \ln\left(\sqrt{\left(cleafh_n(l)\right)^n + 1} + \sqrt{\left(cleafh_n(l)\right)^n - 1}\right) - \ln\sqrt{2} \\
&0 \le l < \eta_n \\
&\quad n = 2, 3, \cdots
\end{aligned} \tag{I1}$$

The proof is as follows：

$$\begin{aligned}
&\frac{d}{dl}\left(\sqrt{\left(cleafh_n(l)\right)^n + 1}\right) = \frac{d}{dl}\left(\left(cleafh_n(l)\right)^n + 1\right)^{\frac{1}{2}} \\
&= \frac{1}{2}\left(\left(cleafh_n(l)\right)^n + 1\right)^{\frac{1}{2}-1} \cdot n\left(cleafh_n(l)\right)^{n-1} \cdot \sqrt{\left(cleafh_n(l)\right)^{2n} - 1} \\
&= \frac{n}{2}\frac{\left(cleafh_n(l)\right)^{n-1}\sqrt{\left(\left(cleafh_n(l)\right)^n + 1\right)\left(\left(cleafh_n(l)\right)^n - 1\right)}}{\sqrt{\left(cleafh_n(l)\right)^n + 1}} \\
&= \frac{n}{2}\left(cleafh_n(l)\right)^{n-1}\sqrt{\left(cleafh_n(l)\right)^n - 1}
\end{aligned} \tag{I2}$$

$$\begin{aligned}
&\frac{d}{dl}\left(\sqrt{\left(cleafh_n(l)\right)^n - 1}\right) = \frac{d}{dl}\left(\left(cleafh_n(l)\right)^n - 1\right)^{\frac{1}{2}} \\
&= \frac{1}{2}\left(\left(cleafh_n(l)\right)^n - 1\right)^{\frac{1}{2}-1} \cdot n\left(cleafh_n(l)\right)^{n-1} \cdot \sqrt{\left(cleafh_n(l)\right)^{2n} - 1} \\
&= \frac{n}{2}\frac{\left(cleafh_n(l)\right)^{n-1}\sqrt{\left(\left(cleafh_n(l)\right)^n + 1\right)\left(\left(cleafh_n(l)\right)^n - 1\right)}}{\sqrt{\left(cleafh_n(l)\right)^n - 1}} \\
&= \frac{n}{2}\left(cleafh_n(l)\right)^{n-1}\sqrt{\left(cleafh_n(l)\right)^n + 1}
\end{aligned} \tag{I3}$$

Using Eq. (I2) and Eq. (I3), the following equation is obtained:

$$\begin{aligned}
&\frac{d}{dl}\ln\left(\sqrt{\left(cleafh_n(l)\right)^n + 1} + \sqrt{\left(cleafh_n(l)\right)^n - 1}\right) \\
&= \frac{\frac{n}{2}\left(cleafh_n(l)\right)^{n-1}\sqrt{\left(cleafh_n(l)\right)^n - 1} + \frac{n}{2}\left(cleafh_n(l)\right)^{n-1}\sqrt{\left(cleafh_n(l)\right)^n + 1}}{\sqrt{\left(cleafh_n(l)\right)^n + 1} + \sqrt{\left(cleafh_n(l)\right)^n - 1}} \\
&= \frac{n}{2}\left(cleafh_n(l)\right)^{n-1}\frac{\sqrt{\left(cleafh_n(l)\right)^n - 1} + \sqrt{\left(cleafh_n(l)\right)^n + 1}}{\sqrt{\left(cleafh_n(l)\right)^n + 1} + \sqrt{\left(cleafh_n(l)\right)^n - 1}} \\
&= \frac{n}{2}\left(cleafh_n(l)\right)^{n-1}
\end{aligned} \tag{I4}$$

In the case *n=1* of Eq. (I1), the following equation is obtained:

$$\begin{aligned}
&\frac{1}{2}\int_0^l \left(cleafh_1(t)\right)^0 dt \\
&= \ln\left(\sqrt{\left(cleafh_1(l)\right)^1 + 1} + \sqrt{\left(cleafh_1(l)\right)^1 - 1}\right) - \ln\sqrt{2}
\end{aligned} \tag{I5}$$

$$\frac{1}{2}\int_0^l dt = \ln\left(\sqrt{cleafh_1(l) + 1} + \sqrt{cleafh_1(l) - 1}\right) - \ln\sqrt{2} \tag{I6}$$

$$\frac{1}{2}l + \ln\sqrt{2} = \ln\left(\sqrt{cleafh_1(l) + 1} + \sqrt{cleafh_1(l) - 1}\right) \tag{I7}$$

$$e^{\frac{1}{2}l + \ln\sqrt{2}} = \sqrt{cleafh_1(l) + 1} + \sqrt{cleafh_1(l) - 1} \tag{I8}$$

Therefore, the following equation is obtained:

$$\sqrt{2}e^{\frac{1}{2}l} = \sqrt{cleafh_1(l) + 1} + \sqrt{cleafh_1(l) - 1} \tag{I10}$$

Using Eq. (16), the above equation represents the following equation:

$$e^l = \cosh(l) + \sinh(l) \tag{I11}$$

Appendix J

The numerical data of the hyperbolic leaf function is summarized in the table 4.

Table 4 Numerical data of hyperbolic leaf function $cleafh_n(l)$
(All results have been rounded to no more than five significant figures)

l	$r\ (=cleafh_n(l))$				
	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
0.0	1.0000	1.0000	1.0000	1.0000	1.0000
0.2	1.0200	1.0408	1.0632	1.0886	1.1193
0.4	1.0810	1.1741	1.3063	1.5978	-1.6710
0.6	1.1854	1.4425	2.2251	-1.4175	-1.0496
0.8	1.3374	1.9702	-2.2494	-1.0574	-1.0104
1.0	1.5430	3.2181	-1.3107	-1.0026	-1.2510
1.2	1.8106	9.0068	-1.0646	-1.1293	1.2736
1.4	2.1508	-11.240	-1.0000	-1.9365	1.0130
1.6	2.5774	-3.4629	-1.0617	1.3008	1.0020
1.8	3.1074	-2.0568	-1.3020	1.0340	1.5862
2.0	3.7621	-1.4842	-2.2016	1.0105	-1.1305
2.2	4.5679	-1.1959	2.2746	1.1822	-1.0001
2.4	5.5569	-1.0505	1.3151	3.2310	-1.1089
2.6	6.7690	-1.0004	1.0661	-1.2181	1.7851
2.8	8.2527	-1.0321	1.0000	-1.0172	1.0557
3.0	10.067	-1.1540	1.0603	-1.0240	1.0081
3.2	12.286	-1.4036	1.2977	-1.2523	1.2303
3.4	14.998	-1.8910	2.1787	2.3356	-1.2987
3.6	18.312	-3.0059	-2.3007	1.1565	-1.0160
3.8	22.361	-7.5141	-1.3196	1.0062	-1.0387
4.0	27.308	14.944	-1.0676	1.0043	-1.5195
4.2	33.350	3.7485	-1.0000	1.3482	1.1425
4.4	40.731	2.1519	-1.0589	-1.7507	1.0005
4.6	49.747	1.5292	-1.2935	-1.1095	1.0992
4.8	60.759	1.2192	-2.1566	-1.0007	-1.9533
5.0	74.209	1.0614	2.3276	-1.0705	-1.0623
5.2	90.638	1.0019	1.3241	-1.4881	-1.0061
5.4	110.70	1.0246	1.0692	1.5051	-1.2115

Note: The value of the hyperbolic leaf function with respect to the inequality $l < 0$ can be calculated by using the characteristic of the even function (Eq. (19)).